Math 520: Fall 2016

Problem Set 2

1. Is a subring of a Noetherian ring Noetherian? Give a proof or counterexample.

2. Let R be Noetherian and let $I \subseteq R$ be an ideal. Show that $\operatorname{Spec} R/I$ is irreducible if and only if $\sqrt{I}$ is prime.

3. Let $f \in R$. Show that $R_f \cong R[X]/(Xf-1)$. Thus, R_f is a finitely-generated R-algebra.

   Solution: Since the natural map $R \to R[X]/(Xf-1)$ maps f to a unit, the universal property of localization gives an extension $\phi : R_f \to R[X]/(Xf-1)$. By mapping X to $1/f \in R_f$, we have a map $R[X] \to R_f$ whose kernel contains $Xf-1$, thereby producing an induced map $\psi : R[X]/(Xf-1) \to R_f$. Checking that $\phi\psi$ and $\psi\phi$ are the identity is routine.

4. Let $S \subset T$ be multiplicative subsets of a ring R. Let $\widetilde{T}$ be the image of T in $S^{-1}R$. Show that $T^{-1}R \cong \widetilde{T}^{-1}(S^{-1}R)$.

5. A ring R is called *reduced* if its nilradical is zero. Show that R is reduced if and only if $R_\mathfrak{p}$ is reduced for every prime ideal $\mathfrak{p}$.

6. Suppose that every prime ideal of R is maximal. Show that $\operatorname{Spec} R$ is a Hausdorff space. *Hint: Reduce to the case where R has trivial nilradical. Given primes $\mathfrak{p}$ and $\mathfrak{q}$, choose $f \in \mathfrak{p} - \mathfrak{q}$, and note that f vanishes in $R_\mathfrak{p}$...*

7. Let A be a Noetherian ring and M a finitely-generated A-module. Show that if $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$ module then M_f is free over A_f for some $f \in A$.

8. If R is a reduced ring, show that f is a zerodivisor if and only if f is contained in some minimal prime.

9. Let A be Noetherian aand M a finitely-generated module. Let $\mu_M : \operatorname{Spec} A \to \mathbb{R}$ be such that $\mu_M(\mathfrak{p}) = \dim(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p})$ (where the dimension is being taken over the field $k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$).

   (a) Show that μ_M is upper-semicontinuous. That is, for all $\mathfrak{p} \in \operatorname{Spec} A$ and $\epsilon > 0$, there is an open neighborhood $U \subset \operatorname{Spec} A$ of $\mathfrak{p}$, such that $\mu_M(\mathfrak{q}) \leq \mu_M(\mathfrak{p}) + \epsilon$ for all $\mathfrak{q} \in U$.

   Solution: Fix $\mathfrak{p} \in \operatorname{Spec} A$. Let $x_1/s_1, \cdots, x_n/s_n$ be a minimal generating set for $M_\mathfrak{p}$. Since the s_i are units, it's clear that $x_1/1, \cdots, x_n/1$ also generate $M_\mathfrak{p}$. Define $\phi : A^n \to M$ such that $e_i \mapsto x_i$. Let C be the cokernel. Since $\phi_\mathfrak{p}$ is surjective, $C_\mathfrak{p} = 0$ and since C is finitely-generated, there is a $f \notin \mathfrak{p}$ such that $C_f = 0$. Thus for all $\mathfrak{q} \in D(f)$, $C_\mathfrak{q} = 0$ by virtue of it being a further localization of C_f. In particular, $\mu_M(\mathfrak{q}) \leq \mu_M(\mathfrak{p})$ for all $\mathfrak{q} \in D(f)$.

(b) If A is reduced and μ_M is constant, show that M is locally free (i.e. $M_\mathfrak{p}$ is free for all $\mathfrak{p}$).

Solution: Consider $M_\mathfrak{p}$ for some prime $\mathfrak{p}$. By taking a minimal generating set, we have a surjection $\psi : A_\mathfrak{p}^n \to M_\mathfrak{p}$. Let K be the kernel. Let $\mathfrak{q}_1, \cdots \mathfrak{q}_k$ be the minimal primes of $A_\mathfrak{p}$. Since A is reduced, $A_{\mathfrak{q}_i}$ is a field. From the sequence

$$0 \to K_{\mathfrak{q}_i} \to A_{\mathfrak{q}_i}^n \xrightarrow{\phi_{\mathfrak{q}_i}} M_{\mathfrak{q}_i} \to 0$$

we see that $\phi_{\mathfrak{q}_i}$ is a surjection of vector spaces. Since $\mu_M(\mathfrak{q}_i) = \mu_M(\mathfrak{p}) = n$, we know that $\phi_{\mathfrak{q}_i}$ must be an isomorphism, meaning that $K_{\mathfrak{q}_i} = 0$. Let $S \subset A_\mathfrak{p}$ be the complement of the union of the $\mathfrak{q}_i$. This is a multiplicative set. The only primes of $S^{-1}(A_\mathfrak{p})$ are the $\mathfrak{q}_i$ Thus $S^{-1}K = 0$ since it vanishes at every prime. As K is finitely-generated, there is an $f \in S$ such that $f \in \operatorname{ann}(K)$. However since f lies outside of every minimal prime of $A_\mathfrak{p}$ and $A_\mathfrak{p}$ is reduced we know that it is a non-zerodivisor on $A_\mathfrak{p}$. As $K \subseteq A_\mathfrak{p}^n$ and $f \cdot K = 0$, we must have $K = 0$.

(c) If A is reduced, show that μ_M is continuous if and only if M is locally free.

Solution: ($\Rightarrow$) μ_M has discrete image in $\mathbb{R}$, so continuity is equivalent to being locally constant. Given $\mathfrak{p} \in \operatorname{Spec} A$, there is some $f \notin \mathfrak{p}$ such that μ_M is constant on $D(f)$. In other words, for every prime $\mathfrak{q}$ of A_f, μ_M is constant (thought of as a function of $\operatorname{Spec} A_f$). The result now follows from the previous problem.

($\Leftarrow$)For any $\mathfrak{p}$, $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$ by hypothesis. Thus, there is some $f \notin \mathfrak{p}$ such that M_f is free over A_f, meaning that μ_M is constant on $D(f)$.

10. Let A be a Noetherian ring and M a finitely generated module. We define the *support* of M via $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} A : M_\mathfrak{p} \neq 0\}$. Show the following:

(a) $\operatorname{Supp} M = V(\operatorname{ann}(M))$ where $\operatorname{ann}(M) = \{a \in A : ax = 0 \text{ for all } x \in M\}$.

(b) If $0 \to M' \to M \to M'' \to 0$ is exact, then $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$.