# Math 520: Fall 2016

Problem Set 3

1. Show that the tensor product of $R$-modules is commutative and associative. That is, deduce natural isomorphisms:
   (a) $M \otimes_R N \cong N \otimes_R M$
   (b) $M \otimes_R (N \otimes_R Q) \cong (M \otimes_R N) \otimes_R Q$.

2. Let $A \to B$ be a ring morphism and let $M$ and $N$ be $A$-modules. Show that
$$(M \otimes_A N) \otimes_A B \cong (M \otimes_A B) \otimes_B (N \otimes_A B)$$
(*Hint: Using (1), this is a one-line proof.*)

3. Let $0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0$ be a **exact** sequence of $R$-modules. Show that the following are equivalent:
   (a) There exists a map $\phi : Q \to N$ such that $g\phi = Id_Q$.
   (b) There exists a map $\psi : N \to M$ such that $\psi f = Id_M$.
   (c) There is an isomorphism $\omega : N \to M \oplus Q$ that fits into a commutative diagram:
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & M \oplus Q & \xrightarrow{\pi} & Q & \longrightarrow & 0 \end{array}$$
   Where $i$ and $\pi$ are the canonical inclusion and projections.

   Such a short exact sequence is called a *split exact sequence*; the maps $\phi$ and $\psi$ are usually called *splittings.*

4. Let $0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0$ be a **split** exact sequence. Show that for any $R$-module $E$ we get an induced exact sequence $0 \to E \otimes_R M \xrightarrow{E\otimes f} E \otimes_R N \xrightarrow{E\otimes g} E \otimes_R Q \to 0$.

5. Let $A$ be a noetherian ring and let $M$ and $N$ be finitely-generated $A$-modules.
   (a) Show that $M_{\mathfrak{p}} = 0$ if and only if $M \otimes_A k(\mathfrak{p}) = 0$. ($k(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is the residue field at $\mathfrak{p}$.)
   (b) Show that $M \otimes_A N$ is a finitely-generated $A$-module.
   (c) Show that $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. (Use part (a) and question (2).)

6. A morophism of rings $A \to B$ is called *faithfully flat* if $B$ is a flat $A$-module and for all $A$-modules $M$, $B \otimes_A M = 0$ if and only if $M = 0$.
   (a) If $A \to B$ is faithfully flat, show that the morphism is injective. Give an example of a flat map $R \to S$ which is not injective. (*Hint: Take $R$ to be a product of rings...*)

(b) Let $A \to B$ be faithfully flat. Let $f : M \to N$ be a map of $A$-modules. Show that $f$ is injective (resp. surjective, an isomorphism) if and only if $B \otimes_A f$ is.

(c) Show that $A \to B$ is faithfully flat if and only if $B$ is flat over $A$ and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

(d) Prove that $A \to A[X]$ is faithfully flat.

7. Let $A$ and $B$ be $R$-algebras. Let $i_A : A \to A \otimes_R B$ be defined via $i_A(a) = a \otimes 1$, and similarly define $i_B$. Show that $A \otimes_R B$ is the coproduct of $A$ and $B$ in the category of $R$-algebras. That is, given any $R$-algebra $C$ with $R$-algebra maps $f_A : A \to C$ and $f_B : B \to C$, there is a unique $\phi : A \otimes_R B \to C$ compatible with the $f_A, f_B, i_A, i_B$.

8. (Tensor-Hom Adjunction) For a pair $M$, $N$ of $R$-modules, we define $\operatorname{Hom}_R(M, N)$ to be set of all $R$-linear maps between them.

   (a) Verify that $\operatorname{Hom}_R(M, N)$ is an $R$-module by declaring that $r \cdot \phi(x) = \phi(r \cdot x)$.

   (b) Deduce a natural isomorphism of $R$-modules: $\operatorname{Hom}_R(M \otimes_R N, P) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, P))$.

9. Show that $E$ is a flat $R$-module if and only if the functor $E \otimes_R -$ preserves injective maps. Using this, show that any free $R$-module is flat.