Math 520: Fall 2016

Problem Set 5

1. Let k be a field and suppose A is a finite-type k-algebra. Prove the following strong form of the Nullstellensatz: $\mathfrak{p} \in \operatorname{Spec} A$ is closed if and only if $k(\mathfrak{p})$ is a finite extension of k. *Hint: Identify $k(\mathfrak{p})$ with the fraction field of $A/\mathfrak{p}$.*

2. Recall that we define affine n-space over k via $\mathbb{A}^n_k := \operatorname{Spec}(k[X_1, \cdots X_n])$.
   (a) Let k be a field of characteristic 0. Let $\mathfrak{p} \in \mathbb{A}^n_k$ be a closed point. By the Nullstellensatz, $k(\mathfrak{p})/k$ is a finite algebraic extension. Let $f : \mathbb{A}^n_{\overline{k}} \to \mathbb{A}^n_k$ be the natural map induced by the natural inclusion of polynomial rings. Show that $|f^{-1}(\mathfrak{p})| = [k(\mathfrak{p}) : k]$.
   (b) Show by example that the conclusion of (a) can fail if k has characteristic $p > 0$.

3. Let $f_1, \cdots, f_m \in \mathbb{Q}[X_1, \cdots, X_n]$. Suppose that the f_i have a common root in $\mathbb{C}^n$. Show that the f_i have a common root in $\overline{\mathbb{Q}}^n$.

4. Let G be a finite group of automorphisms of a ring R. Denote by $R^G = \{x \in R : \sigma(x) = x \forall \sigma \in G\}$ the subring of G-invariants.
   (a) Show that $R^G \to R$ is an integral extension.
   (b) Suppose that A is a Noetherian subring of R such that $A \to R$ is finite-type and $A \subset R^G$. Show that $A \to R^G$ is also finite-type.

5. If $A \subset B$ is an integral extension, show that the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed. Using this, conclude that if $f : R \to C$ is a finite map, then $f^* : \operatorname{Spec} C \to \operatorname{Spec} R$ is closed.

6. Let R be a ring of characteristic $p > 0$ and let $F : R \to R$ be the Frobenius (i.e. $F(x) = x^p$ for all $x \in R$). Show that F induces a homeomorphism of $\operatorname{Spec} R$.

7. (Prime Avoidance) Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be prime ideals of a ring R. Suppose that I is an ideal such that $I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_n$. Then $I \subset \mathfrak{p}_i$ for some i.

8. Let L/K be a finite, Galois extension. Let $e_1, \cdots, e_n \in L$ be a basis for L as a K-vectorspace, and let $\{\sigma_1, \sigma_2, \cdots, \sigma_n\} = \operatorname{Gal}(L/K)$. Let M be the $n \times n$ matrix whose ij entry is $\sigma_i(e_j)$. Show that M is invertible.

9. A Noetherian ring A is called a *Nagata ring* if for all $\mathfrak{p} \in \operatorname{Spec} A$, and every finite extension L of $k(\mathfrak{p})$, the integral closure of $A/\mathfrak{p}$ in L is finite over $A/\mathfrak{p}$.
   (a) For k a field, show that a finite-type k algebra is Nagata.
   (b) Show that a finite-type $\mathbb{Z}$ algebra is Nagata.

10. Show that if A is a Nagata domain, then there exists an $0 \neq f \in A$ such that A_f is normal.