Math 520: Fall 2016

Problem Set 6

1. Let k be a field and suppose $k \to A$ is finite-type. Show that if A is Artinian, then A is a finite-dimensional vectorspace over k.

2. Let A be a ring and suppose that $\phi : A^m \to A^n$ is an injection of free modules.
   (a) Show that if A is Noetherian then $m \leq n$. (Hint: Localize at a minimal prime and appeal to length.)
   (b) Show that $m \leq n$ in general by reducing to the Noetherian case.

3. (a) Show that if R is a domain then for any pair $f, g \neq 0$, there is an exact sequence
   $$0 \to R/fR \xrightarrow{g} R/fgR \to R/gR \to 0$$
   (b) Show that if R is a PID and f has prime factorization $f = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ then $\ell(R/fR) = n_1 + n_2 + \cdots + n_k$.

4. Suppose that $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$ is an exact sequence of finite-length R-modules. Show that $\sum_{i=1}^{n} (-1)^i \ell(M_i) = 0$.

5. Let R be a PID. Show that $G_0(R) \cong \mathbb{Z}$.

6. Let k be a field and let $A = k[X, Y]$ with $\mathfrak{m} = (X, Y)$. Compute $\ell(A/\mathfrak{m}^n)$ as a function of n.

7. (Jordan-Holder Theorem) For a ring R, we can fix $\mathcal{E}$ the full subcategory of finite-length R modules. $K_0(\mathcal{E})$ is the Grothendieck group on $\mathcal{E}$. (Here, it's not particularly important whether or not R is Noetherian – the construction of $K_0(\mathcal{E})$ carries through.)
   (a) If $M \in \mathcal{E}$ show that $\operatorname{Supp}(M)$ is a finite set of closed points in $\operatorname{Spec} R$.
   (b) Show that taking length ℓ defines a function $\ell : K_0(\mathcal{E}) \to \mathbb{Z}$.
   (c) For $\mathfrak{m}$ a maximal ideal, denote by $\mathcal{E}_\mathfrak{m}$ the category of finite-length modules on $R_\mathfrak{m}$. Show that $M \mapsto M \otimes_R R_\mathfrak{m}$ induces a functor $\mathcal{E} \to \mathcal{E}_\mathfrak{m}$ and hence a morphism $K_0(\mathcal{E}) \to K_0(\mathcal{E}_\mathfrak{m})$.
   (d) Let M have composition series $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$. Show that the image of $[M] \in K_0(\mathcal{E})$ under the map $K_0(\mathcal{E}) \to K_0(\mathcal{E}_\mathfrak{m}) \xrightarrow{\ell} \mathbb{Z}$ is precisely the number of times the simple module $R/\mathfrak{m}$ appears as a subquotient in the composition series.