Math 520: Fall 2016

Problem Set 7

1. Let R be a Noetherian ring and let M be an R-module. Show that $r \in R$ is a non-zerodivisor on M if and only if r lies outside of every associated prime $\mathfrak{p} \in \operatorname{Ass}(M)$.

2. Let $M = M_1$ be a Noetherian R-module. Suppose that we have a sequence of **surjections**:
$$M_1 \stackrel{\phi_1}{\to} M_2 \stackrel{\phi_2}{\to} M_3 \stackrel{\phi_3}{\to} \cdots$$
Show that ϕ_i is an isomorphism for all $i >> 0$.

3. Let R be Noetherian and M a finitely-generated R-module. Prove the $\operatorname{Ass}(M)$ is finite. *Hint: Put $M_1 = M$. Let $\mathfrak{p} \in \operatorname{Ass}(M_1)$ and let $M_2 = \operatorname{coker}(R/\mathfrak{p} \to M_1)$... Use the result of question 2.*

4. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely-generated R-module. Show that there exists a non-zerodivisor $t \in \mathfrak{m}$ on M if and only if $\mathfrak{m} \notin \operatorname{Ass}(M)$.

5. Let $(R, \mathfrak{m})$ be a Noetherian ring and M a finitely-generated R-module. Show that the minimal elements of $\operatorname{Ass}(M)$ and $\operatorname{Supp}(M)$ are the same.

6. Let R be a Dedekind domain. Show that if R is a UFD, then it is a PID.

7. Let R be a Dedekind domain with only finitely many prime ideals. Show that R is a PID.

8. Suppose that $A \subseteq B$ are discrete valuation rings with the same fraction field L. Show that $A = B$.

9. Suppose that $R \subseteq \mathbb{Q}$ is a discrete valuation ring. Show that $R = \mathbb{Z}_{(p)}$ for some prime p.

10. Let $L = \mathbb{C}(X)$ be a purely transcendental field extension of $\mathbb{C}$. Determine all discrete valuation rings $(R, \mathfrak{m})$ that contain $\mathbb{C}$. (These valuation rings are in one-to-one correspondence with the closed points of the scheme $\mathbb{P}^1_{\mathbb{C}}$.) *Hint: Consider the cases $X \in R^\times$, $X \in \mathfrak{m}$, and $X \notin R$.*