Math 520: Fall 2016

Problem Set 8

1. Consider the Dedekind domain $R = \mathbb{Z}[\sqrt{-5}]$.
   (a) Show that the ideal $I = (2, 1 + \sqrt{-5})$ is not principal. (Use the norm on $\mathbb{C}$.)
   (b) Compute the prime ideal factorizations of (3), (4), and (5).

2. Let R be a Noetherian domain with fraction field K. Let $J \subset K$ be a fractional ideal. Show that $J^{-1} = \{t \in K : t \cdot J \subset R\}$ is also a fractional ideal

3. Let M be a nonzero ideal of a Dedekind domain R. Show that $MM^{-1} = R$. Conclude that M^{-1} is the inverse of $M \in \mathrm{Pic}(R)$.

4. Let M be an R-module.
   (a) Show M is flat if and only if $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for all primes $\mathfrak{p}$.
   (b) Show that if M is projective then $S^{-1}M$ is projective over $S^{-1}R$ for any multiplicative S.
   (c) Show that if M is projective then it is flat. (This would follow from (a) and (b) if we knew that projective modules over local rings were free. This is a hard theorem of Kaplansky. There is also an elementary proof: If M is projective then it has a split injection into a free module F. Use the split exactness of $0 \to P \to F \to F/P \to 0$ to show that if $M' \to M$ is injective then so is $M' \otimes P \to M \otimes P$.)

5. Let R be a Dedekind domain. We can classify finitely generated modules over R much in the same way as in the case of a PID.
   (a) Show that every locally free R module splits into a direct sum of invertible R modules.
   (b) Show that if M is a torsion module, then $M \cong R/P_1^{n_1} \oplus \cdots \oplus R/P_k^{n_k}$.
   (c) Show that any finitely-generated module M decomposes into $M \cong E \oplus T$ where E is locally free and M is torsion.

6. Let A be a Noetherian normal domain with fraction field K. Show that

$$\ker(K^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(A)) \subset A^\times.$$

7. Let R be a Dedekind domain with fraction field K. Give a direct proof that R is a UFD if and only if $\mathrm{Pic}\, R = 0$. (We will later prove this using the Divisor class group.) *Hint: The only if part is almost immediate. For the other direction, recall that every rank one projective module is isomorphic to a fractional ideal...*

8. (Optional) If R is a Dedekind domain, show that there is an isomorphsm $G_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}\, R$.