Math 520: Fall 2016

Problem Set 9

1. Let A be a Noetherian normal domain with field of fractions A. Let $\operatorname{div} : K^\times \to \operatorname{Div}(A)$ be the divisor map. A divisor $D = \sum a_i[\mathfrak{p}_i]$ is called effective if $a_i \geq 0$ for all i. Fix any divisor $E \in \operatorname{Div}(A)$. Put $M = \{f \in K : \operatorname{div}(f) + E \text{ is effective.}\}$. Show that M is a fractional ideal:
   (a) Show first that M is an A-submodule of K. (Use the additive and multiplicative properties of valuations)
   (b) Write $E = \sum a_i[\mathfrak{p}_i] - \sum b_j[\mathfrak{q}_j]$ with $a_i, b_j \geq 0$. Pick $0 \neq x \in \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_n^{a_n}$.
   (c) If $f \in M$, show that $\operatorname{div}(xf)$ is effective and conclude, using Hartog's Lemma, that $xf \in A$.
2. Let R be a Dedekind domain. Show that every fractional ideal is locally free of rank one.
3. Show that if R is a Dedekind domain then $\operatorname{Pic}(R) \cong \operatorname{CH}^1(R)$. (In class we constructed an injection $\operatorname{Pic} R \to \operatorname{CH}^1(R)$; use Exercise 2 to show it's surjective.)
4. Show that a Noetherian normal domain is a UFD if and only if every height-one prime is principal.
5. Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is a function for which there exists a degree-d polynomial $p \in \mathbb{Q}[t]$ and an equality $f(n) = p(n)$ for all $n >> 0$. Let $F(n) = \sum_{i=1}^{n} f(i)$. Show that $F(n)$ is a polynomial of degree $d+1$ for $n >> 0$.
6. Consider the graded ring $S = \mathbb{C}[X, Y, Z]$ with grading given by degree. Let $f = X^2Y + Z^3 + Y^2Z$ and let $M = S/fS$. Compute $\dim_{\mathbb{C}}(M_n)$ for all n.
7. Suppose that $S = \oplus_{i=0}^{\infty} S_i$ is graded and that $f \in S$ is a zerodivisor. Suppose that $f = f_r + f_{r+1} + \cdots + f_{r+k}$ with each $f_i \in S_i$. Show that each homogeneous piece f_i is also a zerodivisor.