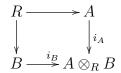
Math 520: Fall 2016

Problem Set 4

1. Let A and B be R-algebras. Consider the pushout square



- (a) If A = R[X], show that $A \otimes_R B \cong B[X]$.
- (b) If $R \to A$ is finite, show that $B \to A \otimes_R B$ is finite.
- (c) If $R \to A$ is flat, show that $B \to A \otimes_R B$ is flat.
- (d) If $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective, show that $\operatorname{Spec}(A \otimes_R B) \to \operatorname{Spec} B$ is surjective.
- 2. Let $A \subset B$ be an integral extension of rings. Let L be an algebraically closed field. Let $\phi: A \to L$ be any morphism. Show that there is an extension $\Phi: B \to L$ by completing the following outline:
 - (a) Let $\mathfrak{p} = \ker \phi$. We have an induced $\overline{\phi}A/\mathfrak{p} \to L$. By the Going-up theorem, there is a $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Show that it suffices to extend $\overline{\phi}$ to B/\mathfrak{q} . Thus, reduce to the case where A and B are domains.
 - (b) Now, reduce to the case where A and B are fields.
 - (c) Use Zorn's Lemma to prove the statement.
- 3. Let $A \subset B$ be an integral extension.
 - (a) Prove that for primes $\mathfrak{q} \subset \mathfrak{q}'$ of B, with $\mathfrak{q} \neq \mathfrak{q}'$, we have $\mathfrak{q} \cap A \neq \mathfrak{q}' \cap A$.
 - (b) Conclude, using the Going-up theorem, that $\dim A = \dim B$.
- 4. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring morphisms.
 - (a) Show that if f and g are finite-type, then gf is finite-type.
 - (b) Show that if gf is finite-type and g is finite then f is finite-type.
- 5. Let A be a finite-type \mathbb{Z} -algebra. Prove that for every maximal ideal \mathfrak{m} of A, A/\mathfrak{m} is a finite field. Hint: Show that the case of A/\mathfrak{m} having characteristic 0 leads to a contradiction by using 4(b).
- 6. Let k be an infinite field. Let $0 \neq f \in k[X_1, \dots, X_n]$.
 - (a) Prove that there are $a_i \in k$ such that $f(a_1, a_2, \dots, a_n) \neq 0$.
 - (b) Prove that there are $b_i \in k$ and an automorphism ϕ of $k[X_1, \dots, X_n]$ mapping $X_i \mapsto b_i$ such that $\phi(f)$ is monic in X_n . Remark: Nagata showed that even if k is finite, there still exists an automorphism of $k[X_1, \dots, X_n]$ that will make f monic. However, it will no longer preserve the degree of f as in (b).

- 7. Let A be a finite-dimensional k-algebra where k is a field.
 - (a) Show that if A is a domain, then it is a field.
 - (b) Show that every prime ideal of A is maximal.
 - (c) Show that the cardinality of Spec A is bounded by $\dim_k A$.
 - (d) If $\psi: B \to C$ is a finite morphism of rings, show that for every $\mathfrak{p} \in \operatorname{Spec} B$, there are only finitely many $\mathfrak{q} \in \operatorname{Spec} C$ such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.