**Math 520: Fall 2016**

Problem Set 4

1. Let $A$ and $B$ be $R$-algebras. Consider the pushout square

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle i_A} \\ B & \xrightarrow{i_B} & A \otimes_R B \end{array}$$

   (a) If $A = R[X]$, show that $A \otimes_R B \cong B[X]$.

   (b) If $R \to A$ is finite, show that $B \to A \otimes_R B$ is finite.

   (c) If $R \to A$ is flat, show that $B \to A \otimes_R B$ is flat.

   (d) If $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective, show that $\operatorname{Spec}(A \otimes_R B) \to \operatorname{Spec} B$ is surjective.

2. Let $A \subset B$ be an integral extension of rings. Let $L$ be an algebraically closed field. Let $\phi : A \to L$ be any morphism. Show that there is an extension $\Phi : B \to L$ by completing the following outline:

   (a) Let $\mathfrak{p} = \ker \phi$. We have an induced $\overline{\phi}A/\mathfrak{p} \to L$. By the Going-up theorem, there is a $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Show that it suffices to extend $\overline{\phi}$ to $B/\mathfrak{q}$. Thus, reduce to the case where $A$ and $B$ are domains.

   (b) Now, reduce to the case where $A$ and $B$ are fields.

   (c) Use Zorn's Lemma to prove the statement.

3. Let $A \subset B$ be an integral extension.

   (a) Prove that for primes $\mathfrak{q} \subset \mathfrak{q}'$ of $B$, with $\mathfrak{q} \neq \mathfrak{q}'$, we have $\mathfrak{q} \cap A \neq \mathfrak{q}' \cap A$.

   (b) Conclude, using the Going-up theorem, that $\dim A = \dim B$.

4. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring morphisms.

   (a) Show that if $f$ and $g$ are finite-type, then $gf$ is finite-type.

   (b) Show that if $gf$ is finite-type and $g$ is finite then $f$ is finite-type.

5. Let $A$ be a finite-type $\mathbb{Z}$-algebra. Prove that for every maximal ideal $\mathfrak{m}$ of $A$, $A/\mathfrak{m}$ is a finite field. *Hint: Show that the case of $A/\mathfrak{m}$ having characteristic $0$ leads to a contradiction by using 4(b).*

6. Let $k$ be an infinite field. Let $0 \neq f \in k[X_1, \cdots, X_n]$.

   (a) Prove that there are $a_i \in k$ such that $f(a_1, a_2, \cdots, a_n) \neq 0$.

   (b) Prove that there are $b_i \in k$ and an automorphism $\phi$ of $k[X_1, \cdots, X_n]$ mapping $X_i \mapsto b_i$ such that $\phi(f)$ is monic in $X_n$. *Remark: Nagata showed that even if $k$ is finite, there still exists an automorphism of $k[X_1, \cdots, X_n]$ that will make $f$ monic. However, it will no longer preserve the degree of $f$ as in (b).*

7. Let $A$ be a finite-dimensional $k$-algebra where $k$ is a field.
   (a) Show that if $A$ is a domain, then it is a field.
   (b) Show that every prime ideal of $A$ is maximal.
   (c) Show that the cardinality of $\operatorname{Spec} A$ is bounded by $\dim_k A$.
   (d) If $\psi : B \to C$ is a finite morphism of rings, show that for every $\mathfrak{p} \in \operatorname{Spec} B$, there are only finitely many $\mathfrak{q} \in \operatorname{Spec} C$ such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.