Notions of smoothness

<u>Def</u>: $X \xrightarrow{f} Y$ morphism of Noetherian schemes
f is <u>smooth</u> if $\forall x \in X$ $\exists U_x \ni x$ and
a factoring

$$\begin{tikzcd}
U_x \subset X \arrow[rr, "f"] \arrow[dr, "\text{ét}"'] & & Y \\
& \mathbb{A}^n_Y \arrow[ur, "p"'] &
\end{tikzcd}$$

ALL schemes Noetherian unless otherwise stated

~~Let's look at the fibres of f:~~

$$\begin{tikzcd}
\cancel{f^{-1}(y) \cap U_x} = U_x \times \operatorname{Spec} k \arrow[r] \arrow[d, "\text{ét}"] & U_x \subset X \arrow[d, "\text{ét}"] \\
\mathbb{A}^n_k \arrow[r] \arrow[d] & \mathbb{A}^n_Y \arrow[d] \\
\operatorname{Spec} k(y) \arrow[r] & Y
\end{tikzcd}$$

From this definition, it's clear that smoothness is stable under base-change.

Let's look at the fibres of f: $y = \operatorname{Spec} k$, $\bar{y} = \operatorname{Spec} \bar{k}$

$$\begin{tikzcd}
X_{\bar{y}} \arrow[r] \arrow[d] & X_y \arrow[r] \arrow[d] & X \arrow[d] \\
\bar{y} \arrow[r] & y \arrow[r] & Y
\end{tikzcd}$$

$X_{\bar{y}}$ is smooth over an algebraically closed field and so is a regular scheme.

$X_{\bar{y}} \to X_y$ is faithfully flat $\Rightarrow$ X_y is regular.

Regularity of geometric fibres $\Rightarrow$ regularity of all fibres.

<u>Theorem</u>: A morphism $f: X \to Y$ is smooth if and only if

1. f flat
2. f is finite ~~type~~ presentation
3. $\forall$ alg closed $\bar{y} = \operatorname{spec} \bar{k}$ The fibre $\bar{y} \times_y X$ is regular.

We want a more flexible definition that relaxes (2).

<u>Def</u>: Let k be a field and let noetherian A be a k-algebra. A is <u>geometrically regular over k</u> if $\forall L/k$ finite

$A \otimes_k L$ is regular.

<u>Rmk</u>: If $A \otimes_k \bar{k}$ were noetherian (e.g. if A/k were finite type), then it would suffice to ~~say~~ check $A \otimes_k \bar{k}$ regular.

b/c $A \otimes_k L \longrightarrow A \otimes_k \bar{k}$ faithfully flat.

<u>Example</u>: $\mathbb{Q} \to \bar{\mathbb{Q}}$ is geometrically regular but $\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ is not noetherian.

~~Def: A morphism of Noeth. schemes $f: X \to Y$ is geometrically regular if~~

~~(1) f flat~~

~~(2) $\forall y \in Y$~~

Def: $A \xrightarrow{f} B$ is geometrically regular if
(1) f is flat
(2) for all $p \in \operatorname{Spec} A$, $B \otimes k(p)$ geom. reg / $k(p)$.

Theorem: (Popescu) Suppose $A \to B$ is geometrically regular. Then B is a direct limit of smooth A-algebras.

RMK: If $p \in \operatorname{Spec} B$ and $B = \varinjlim_{\alpha \in \Gamma} A_\alpha$.

$f_\alpha : A_\alpha \to B$. put $p_\alpha = f_\alpha^{-1}(p)$ and then

$$B_p = \varinjlim (A_\alpha)_{p_\alpha}$$

RMK: Converse of Popescu's theorem is true.

$A \to B$, $B = \varinjlim A_\alpha$. A_α / A smooth.

If $k(p)$ is a residue field of A. ~~B~~ $L / k(p)$ finite

$B \otimes_A L = \varinjlim A_\alpha \otimes_A L$ each $A_\alpha \otimes_A L$ is regular (b/c A_α smooth.

Suffices to show that if R is noetherian and a direct limit of regular rings R_α then R is regular.

WLOG, R, R_α local w/ max ideals m_α, m_α.

$$\operatorname{Sym}(m_\alpha / m_\alpha^2) \xrightarrow{\sim} gr_{m_\alpha} R_\alpha$$

Pass to limits,

$$\boxed{\begin{array}{c}A \to B \\ \text{smooth}\end{array}} \xRightarrow{\qquad} \boxed{\begin{array}{c}A \to B \\ \text{geom.} \\ \text{regular}\end{array}}$$

$$\xLeftarrow{\qquad}$$

(if B/A finite type)

Popescu gives us a means of going from the RHS to the LHS in general.

<u>Example</u>: Let (A, m) be an <u>excellent</u> local ring.

By def, $A \to \hat{A}$ is geometrically regular. so $\hat{A} = \varinjlim A_\alpha$ where A_α / A smooth.

This is why we care about excellent rings in Artin approximation.

Formal <u>Smoothness</u>

<u>Def</u>: Let $A \to B$ be a ring map $I \subset B$ an ideal. We say that $A \to B$ is <u>I-smooth</u> if for all rings C and ideals $J \subset C$ w/ $J^2 = 0$, and diagram

$$\begin{tikzcd} B \arrow[r, "\varphi"] & C/J \\ A \arrow[u] \arrow[r] & C \arrow[u, two heads] \end{tikzcd}$$

w/ $\varphi(I^n) = 0$ for some n.

There is a <u>lift</u> $\tilde{\varphi} : B \to C$.

(saying $\varphi(I^n) = 0$ means that if C/J is discrete φ is continuous for the I-adic topology on B).

If the lift is unique we say $A \to B$ is I-étale.

RMK: If $I \subset I'$ I-smooth $\Rightarrow$ I'-smooth.

When $I = 0$, we shall use the term quasi-smooth quasi-étale.

Examples:

(1)
$$
\begin{tikzcd}
S^{-1}A \arrow[r] & C/J \\
A \arrow[u] \arrow[r, "f"'] & C \arrow[u]
\end{tikzcd}
\qquad J^2 = 0
$$

f maps each $s \in S$ to a unit in C/J and hence a unit in C b/c J is nilpotent.

$A \to S^{-1}A$ quasi-étale by universal property of localization.

(2)
$$
\begin{tikzcd}
A[x] \arrow[r, "\varphi"] \arrow[dr, dashed, "\tilde{\varphi}"] & C/J \\
A \arrow[u] \arrow[r] & C \arrow[u]
\end{tikzcd}
\qquad \varphi(x_i) = \bar{c}_i
$$

Let $\tilde{\varphi}$ map x_i to any lift of $\bar{c}_i$.

$A \to A[x]$ smooth.

③ Let $f \in A[x]$ be monic. Consider

$A \to (A[x]/(f))_{f'}$ Consider

$$\begin{tikzcd}
(A[x]/(f))_{f'} \arrow[r, "\varphi"] & C/J \\
A \arrow[u] \arrow[r] & C \arrow[u, two heads]
\end{tikzcd}$$

say φ maps x to $\bar{c}$ then

$f(c) \in J$. If $\varepsilon \in J$ and

$f(x) = \sum a_n x^n$. observe that

$$f(x+\varepsilon) = \sum a_n \cancel{x^n} (x+\varepsilon)^n$$

$$= \sum a_n \sum_{d=0}^{n} \binom{n}{d} x^{n-d} \varepsilon^d$$

$$= \sum a_n (n x^{n-1} \varepsilon + x^n)$$

$$= \varepsilon f'(x) + f(x).$$

~~Let~~ $f'(c) \in C^\times$ so, let $\varepsilon = f(c) \cdot f'(c)^{-1} \in J$

then $f(c+\varepsilon) = \varepsilon f'(c) + f(c)$

so mapping $x \mapsto c+\varepsilon$ gives the lift $\tilde{\varphi}$.

This lift is unique.

RMK: This shows a finite separable field ext'n L/k is quasi-étale).

Fact: If $k \to L$ is separable then it is quasi-smooth.

RMK: If L/k is finitely generated, we choose a sep. trans. basis so that

$$\underset{\substack{\uparrow \\ \text{polynomial}}}{k \hookrightarrow k[x_1 \cdots x_n]} \subset \underset{\substack{\uparrow \\ \text{localization}}}{k(x_1 \cdots x_n)} \subset \underset{\substack{\uparrow \\ \text{finite sep.}}}{L}$$

Theorem: Let (A, m) be a complete local ring containing a field k. Then $A \simeq A/m[\![x_1 \cdots x_n]\!]/I$.

Pf: Choose a perfect $k_0 \subset k$. then $k_0 \to A/m$ is quasi-smooth.

$$\begin{tikzcd} A/m \arrow[r, "\sim"] & A/m \\ k_0 \arrow[u] \arrow[r] & A/m^2 \arrow[u] \arrow[from=1-1, to=2-2, dashed] \end{tikzcd}$$

Inductively

$$\begin{tikzcd} A/m \arrow[r] \arrow[dr] & A/m^n \\ k_0 \arrow[u] \arrow[r] & A/m^{n+1} \arrow[u] \end{tikzcd} \qquad A/m \to \varprojlim A/m^n = A.$$

Choose generators for m t_i $\qquad (A/m)[\![x_1 \cdots x_n]\!] \twoheadrightarrow A$, $x_i \longmapsto t_i$

Relation between formal smoothness & ordinary smoothness:

Thm: Let (A, m) be a local k-algebra. A/k geometrically regular $\Leftrightarrow$ $k \to A$ m-smooth.

EGA IV 19.7.1

Thm Let $(A, m, k) \to (B, n, L)$ be a local morphism ~~of~~ TFAE
(1) $A \to B$ n-smooth
(2) $A \to B$ flat and $k \to B/mB$ is n-smooth.

Prop: Let $A \to B$ be quasi-smooth. Then $A \to B$ is geometrically regular.

Pf: Let $\mathfrak{q} \in \operatorname{Spec} B$ map to $p \in \operatorname{Spec} A$. Suffices to show $A_p \to B_{\mathfrak{q}}$ is flat and $k(p) \otimes_{A_p} B_{\mathfrak{q}}$ is geom. reg.

~~WLOG, A, B are local.~~

$A_p \longrightarrow A_p \otimes_A B$ quasi-smooth by base change ~~and~~ so composite

$$A_p \to A_p \otimes_A B \underset{\substack{\uparrow \\ \text{localization}}}{\longrightarrow} B_{\mathfrak{q}} \quad \text{is quasi-smooth.}$$

~~WLOG A, B are local~~

Let n be the max ideal of $B_{\mathfrak{q}}$.

$A_p \to B_q$ n-smooth $\Rightarrow$ flatness. and

$k(p) \to B_q \otimes_{A_p} k(p)$ is n-smooth hence geometrically regular. $\square$

Rmk: If $A \to B$ finite type then smooth, geometrically regular, quasi-smooth are all equivalent.

Warning: Geometrically regular $\not\Rightarrow$ Quasi-smooth

Example: ~~(Tanimoto, Some Characterizations of smoothness)~~

$\mathbb{C} \to \mathbb{C}[[x]]$ is geometrically reg. and hence m-smooth, but is not 0-smooth.

If it were 0-smooth then $\Omega_{\mathbb{C}[[x]]/\mathbb{C}}$ would be projective and hence free.

On the other hand, since $\mathbb{C} \xrightarrow{\sim} \mathbb{C}[[x]]/(x)$ we have a split exact sequence

$$0 \to (x)/(x^2) \to \Omega_{\mathbb{C}[[x]]/\mathbb{C}} \otimes \frac{\mathbb{C}[[x]]}{(x)} \to \Omega_{\mathbb{C}/\mathbb{C}} \to 0$$

Note that $\mathrm{rk}\, \Omega_{\mathbb{C}[[x]]/\mathbb{C}} \otimes \mathbb{C}((x)) = \mathrm{tr.deg}_{\mathbb{C}}\, \mathbb{C}((X))$.