11.3 C1

Main Lemma: Let $R \to A \to \Lambda$ w/ A f.t. $/R$. Let $a \in R$ w/

1) $\mathrm{ann}(a) = \mathrm{ann}(a^2)$ in R and Λ

2) a in A is strictly standard.

Let $\bar{R} = R/a^8 R$ etc. Suppose we have $\bar{C}$ ft $/\bar{R}$ w/ $\bar{R} \to \bar{A} \to \bar{C} \to \bar{\Lambda}$

Then $\exists B$ ft $/R$ w/ $R \to A \to B \to \Lambda$ and $\pi^{-1}(h_{\bar{C}}) \subset h_B$

[w/ $\pi : \Lambda \to \bar{\Lambda}$].

Note that from this lemma we have:

resolvability of $\bar{R} \to \bar{A} \to \bar{\Lambda} \supset \bar{P}$ implies resolvability of $R \to A \to \Lambda \supset P$:

Indeed, let $\bar{C}$ resolve $\bar{R} \to \bar{A} \to \bar{\Lambda} \supset \bar{P}$ (ie $\bar{R} \to \bar{A} \to \bar{C} \to \bar{\Lambda}$ w/ $h_{\bar{B}} \subset h_{\bar{C}} \not\subset \bar{P}$ and hence $\pi^{-1}(h_{\bar{C}}) \not\subset P$.)

By the lemma, we have $R \to A \to B \to \Lambda$ w/ $\pi^{-1}(h_{\bar{C}}) \subset h_B$.

$H_{A/R} \bar{A} \subset H_{\bar{A}/\bar{R}}$ (stability of smoothness under base change).

$$\begin{tikzcd} A \arrow[r] \arrow[d, "H_{A/R}"'] & \Lambda \arrow[d, "\pi"] \\ \bar{A} \arrow[r] & \bar{\Lambda} \end{tikzcd}$$

$\pi(H_{A/R}\Lambda) = (H_{A/R}\bar{A})\bar{\Lambda} \subset H_{\bar{A}/\bar{R}}\bar{\Lambda}$ / Lemma

$\Rightarrow h_A \subset \pi^{-1}(h_{\bar{A}}) \subset \pi^{-1}(h_{\bar{C}}) \subset h_B$ $\square$

$\not\subset P$.

The Proof of Main Lemma 11.3 follows from two very technical calculations:

1) Lifting Lemma: Let $R \to \Lambda$ w/ $d \in R$ satisfying $\mathrm{ann}(d) = \mathrm{ann}(d^2)$ in R and Λ. Put $\bar{R} = R/d^2R$ $\tilde{R} = R/dR$. Suppose we have $\bar{R} \to \bar{C} \to \bar{\Lambda}$ $\bar{C}$ ft $/\bar{R}$ then $\exists D$ ft $/R$ w/ $R \to D \to \Lambda$ and a diagram

$$\begin{tikzcd} \bar{R} \arrow[r] \arrow[d] & \bar{C} \arrow[r] \arrow[d] & \bar{\Lambda} \arrow[d, "\pi"] \\ \tilde{R} \arrow[r] & \tilde{D} \arrow[r] & \tilde{\Lambda} \end{tikzcd}$$

flanking vert. maps canonical.

& $\pi^{-1}(h_{\bar{C}}) \subset h_D$.

D) Desingularization Lemma (18.1)

Let $R \to \Lambda$ $d \in R$ satisfying $\operatorname{ann}_\Lambda(d) = \operatorname{ann}_\Lambda(d^2)$. Put $\tilde{R} = R/d^4R$, etc.
Let $R \to A \to \Lambda$ (A ft/R). d in A is strictly standard. Assume $\rho : \tilde{A} \to \tilde{R}$
st

$$\begin{tikzcd} & \tilde{R} \arrow[dl] \arrow[d, equal] & \\ \tilde{A} \arrow[r, two heads, "\rho"'] \arrow[dr] & \tilde{R} \arrow[d] \\ & \tilde{\Lambda} \end{tikzcd}$$

commutes. Put $I = \operatorname{Ann}_R\left[\operatorname{ann}_R(d^2)/\operatorname{ann}_R(d)\right] \subset R$.

Then $\exists B$ st $R \to B \to \Lambda$ and $IB \subset H_{B/R}$.

PROOF OF MAIN LEMMA.

CLAIM: (18.3) Let $R \to \Lambda$, $d \in R$ w/ $\operatorname{ann}(d) = \operatorname{ann}(d^2)$ in R AND Λ
$R \to A \to \Lambda$ w/ $d \in A$ strictly standard. Put $\tilde{R} = R/d^4R$ etc.
Let $R \to D \to \Lambda$ and suppose there is a ~~map~~ factoring $\hat{R} \to \hat{A} \xrightarrow{\hat{f}} \hat{D} \to \hat{\Lambda}$
then there is a ft R alg B $R \to B \to \Lambda$ w/ maps

$$\begin{tikzcd} & A \arrow[d] \arrow[dr] & \\ R \arrow[ur] \arrow[r] \arrow[dr] & B \arrow[r] & \Lambda \\ & D \arrow[u] \arrow[ur] & \end{tikzcd}$$

and $H_{D/R}B \subset H_{B/R}$. $[h_B \subset h_D]$

Pf: We construct B from $E = A \otimes_R D$. $d \in E$ is strictly standard.
This forces a retraction $\tilde{E} \overset{\rho}{\twoheadrightarrow} \tilde{D}$ $(\tilde{a} \otimes \tilde{d} \mapsto f(\tilde{a})\tilde{d})$.

$$\begin{tikzcd} \tilde{E} \arrow[r, two heads, "\rho"] \arrow[dr] & \tilde{D} \arrow[d] \\ & \tilde{\Lambda} \end{tikzcd}$$

We can apply the desingularization lemma to $D \to E \to \Lambda$ to get
B ft $/D$ w/ $D \to E \to B \to \Lambda$ and $\underbrace{\operatorname{Ann}_D\left[\operatorname{ann}_D(d^2)/\operatorname{ann}_D(d)\right]}B \subset H_{B/D}$. Call this ideal I

Fix $c \in H_{D/R}$ $R \to D_c$ is flat so $\operatorname{ann}_R(t) = \operatorname{ann}_{D_c}(t)$ $\forall t \in R$

$\operatorname{ann}_R(d) = \operatorname{ann}_R(d^2) \Rightarrow \operatorname{ann}_{D_c}(d) = \operatorname{ann}_{D_c}(d^2)$

$\Rightarrow I_c = D_c$

$\Rightarrow H_{B_c/D_c} = (1)$.

$\Rightarrow$ B_c smooth / D_c which is R-smooth

$\Rightarrow c \in H_{B/R}$. $\Rightarrow H_{D/R}{}^B \subset H_{D/R}$ $\square$

Pf of 11.3 | We have $R \to A \to \Lambda$

① $\mathrm{ann}(a) = \mathrm{ann}(a^2)$ in R and Λ

② a in A s. standard.

$\bar{R} = R/a^8R$ etc. w/ a ft. $\bar{R}$ alg $\bar{C}$ satisfying $\bar{R} \to \bar{A} \to \bar{C} \to \bar{\Lambda}$

WTF: B st $R \to A \to B \to \Lambda$ w/ $\pi^{-1}(h_{\bar{C}}) \subset h_B$.

Put $d = a^4$. $\bar{R} = R/d^2R$ $\tilde{R} = R/dR$ $= R/a^4R$. we get $R \xrightarrow{ft} D \to \Lambda$ [Lifting Lemma]

$$\begin{array}{ccccc} \bar{R} & \to & \bar{C} & \to & \bar{\Lambda} \\ \downarrow & & f\downarrow\varphi & & \downarrow \\ \tilde{R} & \to & \tilde{D} & \to & \tilde{\Lambda} \end{array} \qquad \pi^{-1}(h_{\bar{C}}) \subset h_D.$$

Look at $\bar{R} \to \bar{A} \to \bar{C} \xrightarrow{\varphi} \tilde{D}$ this induces a map $\tilde{R} \to \tilde{A} \to \tilde{D} \to \tilde{\Lambda}$

Apply CLAIM 18.3 to $R \to D \to \Lambda$ and $R \to A \to \Lambda$ and get B ft $|R$

$$\begin{array}{ccccc} & & A & & \\ & \nearrow & \downarrow & \searrow & \\ R & \to & B & \to & \Lambda \\ & \searrow & \uparrow & \nearrow & \\ & & D & & \end{array} \qquad \text{w/ } h_D \subset h_B$$

$$\pi^{-1}(h_{\bar{C}}) \subset h_D$$

[In the notation of 18.3 d is replaced w/ a]

$\square$.

Sketch of pf of desingularization

Fix a presentation $A = R[x_1 \cdots x_N]/I$

$\varphi(x_i) = \tilde{y}_i \in \hat{R}$ lift to $y_i \in R$. If $q(x) \in I \subset R[x]$ then $q(\tilde{y}_i) = 0$ in $\hat{R}$ $\Rightarrow q(y_i) \in d^4R$.

Say the image of d in A is represented by $P(x) \in \Delta(f_1 \cdots f_r)[(f):I)]$ $\in A$.

in $R[x_1 \cdots x_N]$, $d - P(x) \in I$. $\Rightarrow d - P(y) \in d^4 R$

ie $d - P(y) = d^4 t$ $(t \in R) \Rightarrow P(y) = d\underbrace{(1 - d^3 t)}_{S \in R}$ $S \equiv 1 \bmod d$.