The Bass Quillen Conjecture

Conjecture: Let A regular. $R = A[x_1 \cdots x_n]$ Then every fg. projective R-module P is extended from $\tilde{P} \in A$-mod $(P \simeq \tilde{P} \otimes_R A)$.

Serre Conjecture: If k is a field $R = k[x_1 \cdots x_n]$. Then every fg proj. R-mod is free.

Serre interested in following question: Let $I \subset k[x_1 \cdots x_n]$ ht 2. When is I gen by regular sequence?

RMK if $I = (f, g)$ then

1. I lci (ok)
2. I unmixed
3. $\mathrm{Ext}^1_R(I, R)$ cyclic.

Thm (Serre) Assume $I \subset \underset{=R}{k[x_1 \cdots x_n]}$ ht 2 satisfies 1 2 3 and every rk 2 ~~free~~ proj. R-mod is free. Then I is a complete intersection.

Cor: In $\mathbb{A}^3_k$ (char $k = 0$) every smooth genus 0, 1 is ~~R~~ complete intersection.

(In char p, Cowsik-Nori every curve in $\mathbb{A}^n_k$ is C.I.)

Quillen's Proof of Serre's Conjecture:

Thm (Quillen): $R = A[x]$ A Noetherian. M fg R-mod. If M_m is extended from an A_m module, (for all $m \in \max\operatorname{spec} A$) then M is extended.

Cor: R ~~is~~ $= A[x]$ Let P be fg Projective R-module. If P_f is free R_f for some monic $f \in A[x]$, then P is free.

Proof (Assuming Quillen's Thm).

$R = k[x_1 \cdots x_n]$ Induct on n.

$n=1$ case is trivial b/c R a pid.

Assume true for $n-1$. $S = k[x_n] \setminus \{0\}$.

$S^{-1}R = k(x_n)[x_1 \cdots x_{n-1}]$ $S^{-1}P$ is free by induction.

Since P is fg, there is $f \in S$ such that P_f is free over R_f.
we can take f monic □.

Suslin's Idea:

Def: A noetherian. A sequence ~~sequence~~ $(a_1 \cdots a_n)$ is called a unimodular row if there $v_1 \cdots v_n \in A$ st
$\sum a_i v_i = 1$.

Thm: (Hilbert) $R = k[x_1 \cdots x_m]$ every fg projective R module is stably free.

Prop: TFAE

① Every fg stably free ~~mod~~ A-mod is free.

② Each unimodular row can be completed into an invertible matrix. [ie. $a_1 \cdots a_n$ is a row in some $n \times n$ invertible matrix].

Suslin: Over $R = k[x_1 \cdots x_m]$ each unimodular row can be completed to an invertible matrix.

When $n=2$ (a_1, a_2) unimodular $\sum a_i v_i = 1$

$$\begin{bmatrix} a_1 & a_2 \\ v_1 & -v_2 \end{bmatrix}$$

When $n=3$, this fails for some regular rings

$$R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$$

$$0 \to P \to R^3 \xrightarrow{(x,y,z)} R \to 0$$

P is not free (tangent bundle to S^2).

<u>Thm</u> (Lindel): Let A be a regular local ring ess. f.t / k a field. Then every f.g. ~~module~~ $R = A[x_1, \dots x_n]$ ~~are~~ projective module is ~~free~~ extended.

Popescu's <u>Generalization</u>

~~By~~ ~~Take~~

Let R be regular local containing a field. Then all f.g. projective R-modules over the polynomial ring $R[x_1 \cdots x_n]$ are free.

<u>Pf</u>: Let $\mathbb{F} \subset R$ be a prime field.

$\mathbb{F} \to R$ geometrically regular (as $\mathbb{F}$ is perfect).

$R = \varinjlim_{\alpha} A_\alpha$ where A_α is an essentially smooth local $\mathbb{F}$-algebra. so $R[x_1 \cdots x_n] = \varinjlim_{\alpha} A_\alpha[x_1 \cdots x_n]$

Let P be a projective module $/ R[x_1 \cdots x_n] = S$,

Now
$$\begin{tikzcd} P \oplus Q \arrow[r, phantom, "\cong"] \arrow[d] & S^m \arrow[d, "M"] \\ P \oplus Q \arrow[r, phantom, "\cong"] & S^m \end{tikzcd} \qquad M^2 = M$$

Projective modules give idempotent matrices and vice versa.
Since $M^2 = M$ involves only finitely many relations, we can choose $A_\alpha[x_1 \cdots x_n]$ so that it contains all of them.
So P is defined over $A_\alpha[x_1 \cdots x_n]$ and hence is extended from A_α etc.

<u>Question</u>: (Murthy) $I \subset k[x_1 \cdots x_n]$ is $\mu(I) = \mu(I/I^2)$.

<u>Thm</u>: (Fasel) Assume k infinite, perfect. char $\neq 2$.
$I \subset k[x_1 \cdots x_n]$ then $\mu(I) = \mu(I/I^2)$.