ARTIN'S APPROXIMATION THEOREM

① Background:

Thm (Artin, 1968): k valued field of char 0. System of equations

$$\begin{cases} f_1(x_1 \cdots x_m, y_1 \cdots y_n) = 0 \\ \vdots \\ f_t(\cdots) = 0 \end{cases} \quad \text{where } f_i \in k\{\underline{x}, \underline{y}\}.$$

conv. power series.

Assume that $\overline{y} = (y_1(\underline{x}), \cdots y_n(\underline{x})) \in K[[x]]$.

is a solution to $\underline{f} = (f_1 \cdots f_t)$. Then $\exists\, \tilde{y}(x) \in k\{x\}$.

such that $\underline{f}(\underline{x}, \tilde{y}(\underline{x})) = 0$

And $y \equiv \hat{y}$ mod. m^c some c.

Artin later asked whether this is true for systems of polynomial equations for an excellent, henselian local ring.

Without the excellence hypothesis, the conclusion is false:

EXAMPLE: (cf Nagata p 206).

char $k = p$. $[k : k^p] = \infty$ $R = k^p[[x]][k]$

Check that $\hat{R} = K[[X]]$. Then $R \to \hat{R}$ not geom. regular (separability problems).

→ Turns out that approximation fails for such a ring.

Theorem: (Popescu 1986). Artin approximation holds for polynomial systems w/ coeff in an excellent henselian local ring R.

Pf: $A = R[x_1 \cdots x_n]/(f_1 \cdots f_t)$. ~~A solution is just a map $A \to \hat{R}$.~~ (R-alg)

To say that we have a root in $\hat{R}$ says exactly that $\exists$ map $A \to \hat{R}$ in R-alg.

But we have $R \to A \to \hat{R}$ w/ $R \to \hat{R}$ geom reg and $R \to A$ f.t.

Popescu's thm says that there is a smooth R-alg, B w/

$$R \to A \to B \to \hat{R}.$$

Let q be the inverse image of $m_{\hat{R}}$ in B. $C := B_q$.

$R \to C$ is essentially smooth; by the structure theorem for essentially smooth morphisms

$$C = \left(\frac{R[T_1, \dots T_m, y]}{f(T_1 \dots T_m, y)} \right)_p \quad \text{where } \frac{\partial f}{\partial y} \notin p. \ f \text{ monic in } y.$$

We can find $g \in R[T_1 \dots T_m, y]$ (cf 3.8 in Swan) that $R \to C$ factors through $D = \left(\frac{R[T_1 \dots T_m, y]}{f(T_1 \dots T_m, y)} \right)_g$ [w/ g multiple of $\frac{\partial f}{\partial y}$].

~~Pick $\underline{a} = (a_1, \dots a_n) \in \hat{R}^n$ solution of ...~~

$$D \to \hat{R} \text{ given by } \begin{matrix} T_i \mapsto a_i \\ y \mapsto b \end{matrix} \text{ st } f(\underline{a}, \underline{b}) = 0. \ \bar{g}(a,b) \neq 0.$$

Pick $a' = (a_1' \cdots a_n') \in R^n$ $b' \in R$ st $\bar{a}' = \bar{a}$, $\bar{b}' = \bar{b}$.

$g(\bar{a}, \bar{b}) \neq 0$ by hypothesis.

$q(y) := f(a', y) \in R[y]$. $\bar{q}(\bar{b}') = 0$ $\bar{q}'(\bar{b}') \neq 0$. b/c $q' = \frac{\partial f}{\partial y}(a', y)$ divides g.

$\Rightarrow$ $\bar{b}'$ not a repeated root of $\bar{q}$, so ~~by Hensel's lemma~~
since R is henselian, we can find $b'' \in R$ st.

$$q(b'') = 0 = f(a', b'')$$

Defining $T_i \mapsto a_i'$ $y \mapsto b''$ gives the map.

$D \to R$, so we have

$R \to A \to D \to R$ as desired. $\square$