The Gersten Complex:

Let X be regular, Noetherian. Let $X^{(p)} = \{$irreducible, reduced subschemes of codim $p\}$.

$$0 \to G_p(X) \xrightarrow{d_1} \bigoplus_{x \in X^0} K_p(k(x)) \xrightarrow{d_1} \bigoplus_{x \in X^{(1)}} K_{p-1}(k(x)) \to \cdots \to \bigoplus_{x \in X^{(p)}} K_0(k(x)) \to 0.$$

~~4) Where does~~

Conjecture: If X is a regular local scheme this complex is exact.

1) Where does this complex come from?
2) Who cares? if it's exact?
3) Why is the conjecture true for X the local ring of a smooth variety.

Blackboxes from Quillen K-theory:

~~Notherian~~.

$\mathcal{A}$ abelian cat. $\mathcal{E} \subset \mathcal{A}$ full subcategory closed under extensions - i.e. $\mathcal{E}$ is an "exact category"

K_i are functors from $\left\{\begin{matrix}\text{exact categories w/}\\ \text{exact functors}\end{matrix}\right\} \to$ abelian Grps.

such that $K_0(\mathcal{E}) =$ Grothendieck group on $\mathcal{E}$.

Let X be noetherian. We shall be interested in the following

$\mathcal{M}(X) =$ coherent $\mathcal{O}_X$-modules

$\mathcal{M}^p(X) =$ coherent $\mathcal{O}_X$-modules supported in codim $\geq p$.

$\mathcal{P}(X) =$ locally-free ~~mo~~ $\mathcal{O}_X$-modules.

We write $G_i(X) := K_i(\mathcal{M}(X))$

(Resolution) $K_i(X) = K_i(P(X))$.

FACT 1: The natural inclusion $P(X) \subset \mathcal{M}(X)$ induces $K_p(X) \to G_p(X)$. and when X is regular, this map is an isomorphism

When $X = \operatorname{Spec} A$, each $M \in \mathcal{M}(X)$ has a finite resolution by projectives $P_\bullet \to M$. and the map

$$K_0(A) \to G_0(A) \quad \text{has a section}$$

$$\text{\small s}$$

$$s([M]) = \sum_i (-1)^i [P_i].$$

FACT 2: (Localization). Each $M^{p+1}(X)$ embeds into $\mathcal{M}^p(X)$ as a full subcategory. In fact, it is a "Serre subcategory" in that it is closed under extensions, ~~subobjects~~, and cokernels.

We can form the quotient category $\mathcal{M}^p(X)/\mathcal{M}^{p+1}(X) =: \mathcal{C}^p(X)$ which is abelian.

$ob(\mathcal{C}^p(X)) = ob(\mathcal{M}^p(X))$

$\operatorname{Hom}_{\mathcal{C}^p(X)}(A,B) = \varinjlim_{(A',B')} \operatorname{Hom}_{\mathcal{M}^p}(A', B/B')$ [all objects of $\mathcal{M}^{p+1}(X)$ identified w/ 0]

$A/A' \in \mathcal{M}^{p+1}$, $B' \in \mathcal{M}^{p+1}$ $A' \subset A$, $B' \subset B$

The exact sequence of functors

$$\mathcal{M}^{p+1}(X) \subset \mathcal{M}^p(X) \xrightarrow{\pi} \mathcal{C}^p(X).$$

gives LES:

$$K_i(\mathcal{M}^p(X)) \to K_i(\mathcal{C}^p(X)) \xrightarrow{\partial} K_{i-1}(\mathcal{M}^{p+1}(X)).$$

FACT 3:

$$\mathcal{C}^p(X) := \mathcal{M}^p(X) / \mathcal{M}^{p+1}(X) = \coprod_{x \in X^p} \left\{ \begin{array}{c} \text{artinian } \mathcal{O}_{X,x} \\ \text{modules} \end{array} \right\}.$$

K-theory commutes w/ ~~limits~~ direct limits

$$K_i(\mathcal{C}^p(X)) \cong \bigoplus_{x \in X^p} K_i(\text{artinian } \mathcal{O}_{X,x}\text{-mod})$$

K-theory is blind to infinitesimal thickenings (devissage) ⟶

$$\cong \bigoplus_{x \in X^p} K_i(k(x)\text{-modules})$$

$$\simeq \bigoplus_{x \in X^p} K_i(k(x))$$

We construct the Gersten Complex by assembling localization sequences:

$$\begin{tikzcd}
 & & & K_{i-1}(\mathcal{M}^{p+2}(X)) \arrow[d] & \\
K_i(\mathcal{M}^{p+1}(X)) \arrow[r] & K_i(\mathcal{M}^p(X)) \arrow[r] & \bigoplus_{x \in X^p} K_i(k(x)) \arrow[r, "\partial"] \arrow[dr, dotted, "d = \pi\partial"'] & K_{i-1}(\mathcal{M}^{p+1}(X)) \arrow[r] \arrow[d, "\pi"] & \cdots \\
 & & & \bigoplus_{x \in X^{p+1}} K_{i-1}(k(x)) \arrow[d] & \\
 & & & \vdots &
\end{tikzcd}$$

RMK: Together, all of the localization sequences give the data of an exact couple; the Gersten complex is obtained from the d_1-differential of the associated spectral sequence.

<u>Prop</u>: The Gersten complexes are exact (for all p) iff $\mathcal{M}^{p+1}(X) \to \mathcal{M}^p(X)$ induces the 0-map after applying K_*.

Significance of Gersten's Complex.

Let X be regular. Define

$\mathcal{G}_{i,X}$ to be sheaf associated to $U \mapsto G_i(U)$.

$\mathcal{K}_{i,X}$ ——— $\|$ ——— $U \mapsto K_i(U)$.

<u>Thm</u> (Bloch's Formula) Suppose every local ring of X satisfies Gersten's conjecture. Then

$$H^p(X, \mathcal{G}_{p,X}) = H^p(X, \mathcal{K}_{p,X}) = CH^p(X).$$

<u>Proof</u>: We "sheafify" the complex:

Consider $i_x : \operatorname{Spec}(\underbrace{k(x)}_{S_x}) \to X$ and the complex.

$$0 \to \mathcal{G}_{p,X} \to \bigoplus_{x \in X^0} (i_x)_*(\mathcal{K}_{p,S_x}) \to \bigoplus_{x \in X^1} (i_x)_*(\mathcal{K}_{p-1,S_x}) \to \cdots \quad (\star)$$

Localizing at some $y \in X$ gives the Gersten complex for $\operatorname{Spec}(\mathcal{O}_{X,y})$, which is exact by hypothesis.

Hence $(\star)$ is a flabby resolution of $\mathcal{G}_{p,X}$. so

$$H^p(X, \mathcal{G}_{p,X}) = \operatorname{coker}\left(\Gamma\left(X, \bigoplus_{x \in X^{p-1}} (i_x)_*(\mathcal{K}_{1,S_x})\right) \xrightarrow{d} \Gamma\left(X, \bigoplus_{x \in X^p} (i_x)_* K_{0,S_x}\right)\right)$$

$$= \operatorname{coker}\left(\bigoplus_{x \in X^{p-1}} K_1(k(x)) \xrightarrow{d} \bigoplus_{x \in X^p} K_0(k(x))\right).$$

~~Now, $K_1(k(x)) \cong k(x)$~~

If F is a field,

$$K_1(F) = F^\times$$

$$K_0(F) = \mathbb{Z} \quad \text{(all } F\text{-modules free)}.$$ Natural identification

$$\begin{array}{ccc} \bigoplus_{x \in X^{p-1}} K_1(k(x)) & \xrightarrow{d} & \bigoplus_{x \in X^p} K_0(x) \\ \| & & \| \\ \bigoplus_{x \in X^{p-1}} k(x)^\times & \xrightarrow{\text{div}} & Z^p(X) \end{array}$$

codim p cycles

$$H^p(X, \mathcal{G}_{p,X}) = \text{Coker}(\text{div}) = CH^p(X). \qquad \square$$

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③ Proof of Gersten's Conjecture in the Geometric Case.

A smooth $/k$ $\quad R = A_{\mathfrak{p}}$. WLOG A domain.

CLAIM: $\mathcal{M}^{p+1}(R) \subset \mathcal{M}^p(R)$ induces 0 on K-theory.

Reduction: $\mathcal{M}^q(R) = \varinjlim_{f \notin \mathfrak{p}} \mathcal{M}^q(A_f)$. Suffices to show that $\mathcal{M}^{p+1}(A_f) \to \mathcal{M}^p(R)$ induces 0 on K-theory.

[b/c $\varinjlim_{f \notin \mathfrak{p}} K_*(A_f) = K_*(R)$].

A_f smooth $/k$ might as well replace A_f by A.

$$\mathcal{M}^{p+1}(A) = \varinjlim_{t \neq 0} \mathcal{M}^p(A/tA).$$

So suffices to show $\mathcal{M}^P(A/tA) \to \mathcal{M}^P(R)$ induces 0 on K-thy.
We shall find $g \in A$ $g \notin p$. such that $\mathcal{M}^P(A/tA) \xrightarrow{j} \mathcal{M}^P(A_g)$, $M \mapsto M_g$ ⊛
induces 0 on K-theory [NB: $\mathcal{M}^P(A/tA) \to \mathcal{M}^P(R)$ factors thru $\mathcal{M}^P(A_g)$]

<u>Quillen's Moving Lemma</u>: Let A be a smooth k algebra. (domain). of dimension r. Let $t \in A$ nonzero, fix $p \in \operatorname{Spec} A$. Then there is $B = k[x_1 \cdots x_{r-1}]$ st.

1. $B \to A$ is smooth at p.
2. $B \to A/tA$ is finite.

RMK: If $k = \bar{k}$, then we can consider all projections

$$\begin{tikzcd} & \mathbb{A}^N \arrow[d] \\ \operatorname{Spec} A \arrow[ur, hook] \arrow[r] & \mathbb{A}^{r-1}_k \end{tikzcd}$$

Bertini ⇒ generic projection smooth at p.
NNL ⇒ generic projection finite along $\operatorname{Spec}(A/tA)$.

<u>Proof of ⊛</u>: Put $B' = A/tA$.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\text{finite}"'] & A \arrow[d] \\ B' \arrow[r] & A \otimes_B B' = A' \end{tikzcd}$$

Since the smooth locus of A/B contains p and is <u>open</u> there is $f \in A$ st. A_f is B-smooth

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & A_f \arrow[d] \\ B' \arrow[r] & A' \arrow[r] & A'_f \end{tikzcd}$$

$\Longrightarrow A'_f$ smooth $/ B'$.

Finitely many primes $q_1, \cdots q_n \in \operatorname{Spec} A'$ lying over p, all contained in $\operatorname{Spec} A'_f$ ⇒ $B' \to A'$ smooth at each q_i.

Now, $B' \to A'$ retracts onto B'. as $A' = B' \otimes_B A \twoheadrightarrow B'$
$a_1 \otimes a_2 \mapsto a_1 a_2$

FACT: A section of a smooth morphism is ~~[illegible]~~ a local complete intersection. So $B'_{q_i} \to A'_{q_i} \twoheadrightarrow B'_{q_i}$ means that $\operatorname{Ker}(A'_{q_i} \twoheadrightarrow B'_{q_i})$ is ~~[illegible]~~ defined by a reg. sequence of length 1 (for dimensional reasons).

Put $S = \bigcup q_i \subset A'$. $S^{-1}A'$ semilocal. and

$S^{-1}\operatorname{Ker}(A' \twoheadrightarrow B')$ is locally free of rank 1. over $S^{-1}A' = A' \otimes_A A_p$ and hence must be trivial, ~~[illegible]~~

As $\operatorname{Ker}(A' \twoheadrightarrow B')$ is finite over A, there is a $g \in A$ such that

$$\operatorname{Ker}(A' \twoheadrightarrow B') \otimes_A A_g \simeq A'_g$$

Get $0 \to A'_g \to A'_g \to B'_g \to 0$ sequence in f.g. A-mod.
Replace g by gf to ensure that A'_g / B' is smooth.
Given a module $M \in \mathcal{M}^P(A/tA) = \mathcal{M}^P(B')$. Then $M \mapsto M \otimes_{B'} A'_g$ is exact. and defines $F: \mathcal{M}^P(B') \to \mathcal{M}^P(A_g)$ as A'_g / A_g finite.
a functor.

OTOH, $\operatorname{Tor}_1^{B'}(M, B_g) = 0$, so

$$\begin{array}{ccccccc} & & & & M_g & & \\ & & & & \| & & \\ 0 \to A'_g \otimes_{B'} M & \to & A'_g \otimes_{B'} M & \to & B'_g \otimes_{B'} M & \to & 0 \\ \| & & \| & & & & \\ F(M) & & F(M). & & & & \end{array}$$

If $j: \mathcal{M}^P(B') \to \mathcal{M}^P(A_g)$ $M \mapsto M_g$ we have an exact sequence of functors $0 \to F \to F \to j \to 0$ ~~[illegible]~~. from $\mathcal{M}^P(B')$ to $\mathcal{M}^P(A_g)$.

$$K_* F = K_* F + K_* j \Rightarrow K_* j = 0.$$