

### The Gersten Complex:

Let  $X$  be regular, Noetherian. Let  $X^{(p)} = \{ \text{irreducible, reduced subschemes of codim } p \}$ .

$$0 \rightarrow G_p(X) \xrightarrow{d_1} \bigoplus_{x \in X^{(0)}} K_p(k(x)) \xrightarrow{d_2} \bigoplus_{x \in X^{(1)}} K_{p-1}(k(x)) \rightarrow \dots \rightarrow \bigoplus_{x \in X^{(p)}} K_0(k(x)) \rightarrow 0.$$

~~What does~~

Conjecture: If  $X$  is a regular local scheme this complex is exact.

- ① Where does this complex come from?
- ② Who cares if it's exact?
- ③ Why is the conjecture true for  $X$  the local ring of a smooth variety.

### Blackboxes from Quillen K-theory:

~~Noetherian~~

A abelian cat.  $\mathcal{E} \subset \mathcal{A}$  Full subcategory closed under extensions. — i.e.  $\mathcal{E}$  is an "exact category"

$K_i$  are functors from  $\{ \text{exact categories w/ exact functors} \} \rightarrow \text{Abelian Grps}$ .

Such that  $K_0(\mathcal{E}) = \text{Grothendieck group on } \mathcal{E}$ .

Let  $X$  be noetherian. We shall be interested in the following

$\mathcal{M}(X) = \text{Coherent } \mathcal{O}_X\text{-modules}$

$\mathcal{M}^p(X) = \text{Coherent } \mathcal{O}_X\text{-modules supported in codim } \geq p.$

$\mathcal{P}(X) = \text{locally-free } \mathcal{O}_X\text{-modules.}$

~~Write~~ We write  $G_i(X) := K_i(\mathcal{M}(X))$

(Resolution)  $K_i(X) = K_i(\mathcal{P}(X))$ .

FACT 1: ~~then~~ The natural inclusion  $\mathcal{P}(X) \subset \mathcal{M}(X)$  induces  $K_P(X) \rightarrow G_P(X)$ . and when  $X$  is regular, this map is an isomorphism.

When  $X = \text{Spec } A$ , each  $M \in \mathcal{M}(X)$  has a finite resolution by projectives  $P_\bullet \rightarrow M$ . and the map

$K_0(A) \rightarrow G_0(A)$  has a section

$$s([M]) = \sum_i (-1)^i [P_i].$$

FACT 2: (Localization). Each  $\mathcal{M}^{P+1}(X)$  embeds into  $\mathcal{M}^P(X)$  as a full subcategory. In fact, it is a "Serre Subcategory" in that it is closed under extensions, subobjects, and cokernels.

We can form the quotient category  $\mathcal{M}^P(X) / \mathcal{M}^{P+1}(X) =: \mathcal{E}^P(X)$  which is abelian.

$$\text{ob}(\mathcal{E}^P(X)) = \text{ob}(\mathcal{M}^P(X))$$

$$\text{Hom}_{\mathcal{E}^P(X)}(A, B) = \varinjlim_{(A', B')} \text{Hom}_{\mathcal{M}^P(X)}(A', B'/B') \quad [\text{all objects of } \mathcal{M}^{P+1}(X) \text{ identified w/ } 0]$$

$A/A' \in \mathcal{M}^{P+1}, B'/B' \in \mathcal{M}^{P+1} \implies A' \subset A, B' \subset B$

The exact sequence of functors ~~gives~~

$$\mathcal{M}^{P+1}(X) \subset \mathcal{M}^P(X) \xrightarrow{\pi} \mathcal{E}^P(X).$$

gives LES:

$$K_i(\mathcal{M}^P(X)) \rightarrow K_i(\mathcal{E}^P(X)) \xrightarrow{\partial} K_{i-1}(\mathcal{M}^{P+1}(X)).$$

FACT 3:

$$\mathcal{E}^p(x) := \mathcal{U}^p(x) / \mathcal{U}^{p+1}(x) = \coprod_{x \in X^p} \{ \text{artinian } \mathcal{O}_{x,x} \text{ modules} \}$$

K-theory commutes w/ ~~finite~~ direct limits

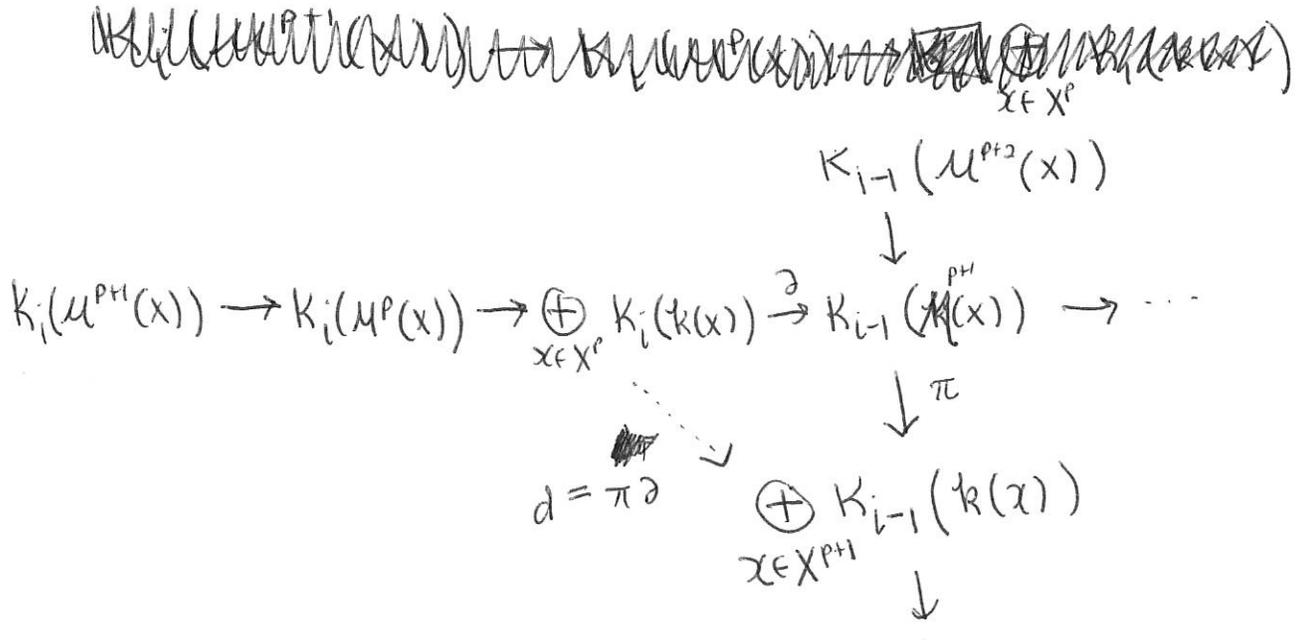
$$K_i(\mathcal{E}^p(x)) \cong \bigoplus_{x \in X^p} K_i(\text{artinian } \mathcal{O}_{x,x}\text{-mod})$$

K-theory is blind to infinitesimal thickenings (devisage)

$$\cong \bigoplus_{x \in X^p} K_i(k(x)\text{-modules})$$

$$\cong \bigoplus_{x \in X^p} K_i(k(x))$$

We construct the Gersten Complex by assembling localization sequences:



RMK: Together, all of the localization sequences give the data of an exact couple; the Gersten complex is obtained from the  $d_1$ -differential of the associated Spectral Sequence.

Prop: The Gersten complexes are exact (for all  $p$ ) iff  $\mathcal{M}^{p+1}(X) \rightarrow \mathcal{M}^p(X)$  induces the 0-map after applying  $K_*$ .

Significance of Gersten's Complex.

Let  $X$  be regular. Define

$\mathcal{G}_{i,X}$  to be sheaf associated to  $U \mapsto G_i(\mathcal{K})$ .

$K_{i,X} \quad \parallel \quad U \mapsto K_i(U)$ .

Thm (Bloch's Formula) Suppose every local ring of  $X$  satisfies Gersten's conjecture. Then

$$H^p(X, \mathcal{G}_{p,X}) = H^p(X, K_{p,X}) = CH^p(X).$$

Proof: We "sheafify" the complex:

Consider  $i_x: \text{Spec}(k(x)) \rightarrow X$  and the complex.

$$0 \rightarrow \mathcal{G}_{p,X} \rightarrow \bigoplus_{x \in X^0} (i_x)_* (K_{p,S_x}) \rightarrow \bigoplus_{x \in X^1} (i_x)_* (K_{p,S_x}) \rightarrow \dots \quad (\star)$$

Localizing at some  $y \in X$  gives the Gersten complex for  $\text{Spec}(U_{X,y})$ , which is exact by hypothesis.

Hence  $(\star)$  is a flabby resolution of  $\mathcal{G}_{p,X}$ . so

$$\begin{aligned} H^p(X, \mathcal{G}_{p,X}) &= \text{coker} \left( \Gamma(X, \bigoplus_{x \in X^{p-1}} (i_x)_* (K_{1,S_x})) \xrightarrow{d} \Gamma(X, \bigoplus_{x \in X^p} (i_x)_* (K_{0,S_x})) \right) \\ &= \text{coker} \left( \bigoplus_{x \in X^{p-1}} K_1(k(x)) \xrightarrow{d} \bigoplus_{x \in X^p} K_0(k(x)) \right). \end{aligned}$$

~~not  $K_1(k(x)) \cong k(x)$~~

If  $F$  is a field,

$$K_1(F) = F^\times$$

$$K_0(F) = \mathbb{Z} \quad (\text{all } F\text{-modules free}). \quad \text{Natural identification}$$

$$\begin{array}{ccc} \bigoplus_{x \in X^{p-1}} K_1(k(x)) & \xrightarrow{d} & \bigoplus_{x \in X^p} K_0(x) \\ \parallel & & \parallel \\ \bigoplus_{x \in X^{p-1}} k(x)^\times & \xrightarrow{\text{div}} & \mathbb{Z}_{(X)}^p \end{array} \quad \leftarrow \text{codim } p \text{ cycles}$$

$$H^p(\mathbb{A}^n, \mathcal{G}_{p,X}) = \text{Coker}(\text{div}) = CH^p(X). \quad \square$$

### ③ Proof of Gersten's Conjecture in the Geometric Case.

$A$  smooth /  $k$   $R = A_p$ . WLOG  $A$  domain.

CLAIM:  $\mathcal{U}^{p+1}(R) \subset \mathcal{U}^p(R)$  induces 0 on  $K$ -theory.

Reduction:  $\mathcal{U}^q(R) = \varinjlim_{f \neq p} \mathcal{U}^q(A_f)$ . Suffices to show

that  $\mathcal{U}^{p+1}(A_f) \rightarrow \mathcal{U}^p(R)$  induces 0 on  $K$ -theory.  
 [b/c  $\varinjlim_{f \neq p} K_*(A_f) = K_*(R)$ ].

$A_f$  smooth /  $k$  might as well replace  $A_f$  by  $A$ .

$$\mathcal{U}^{p+1}(A) = \varinjlim_{t \neq 0} \mathcal{U}^p(A/tA).$$

So suffices to show  $\mathcal{M}^P(A/tA) \rightarrow \mathcal{M}^P(R)$  induces 0 on K-thy.

We shall find  $g \in A, g \notin p$  such that  $\mathcal{M}^P(A/tA) \xrightarrow{f} \mathcal{M}^P(A_g)$  induces 0 on K-theory [NB:  $\mathcal{M}^P(A/tA) \rightarrow \mathcal{M}^P(R)$  factors thru  $\mathcal{M}^P(A_g)$ ]

Quillen's Moving Lemma: Let  $A$  be a smooth  $k$  algebra (domain) of dimension  $r$ . Let  $t \in A$  nonzero. Then there ~~are~~  $p \in \text{Spec } A$  algebra is  $B = k[x_1, \dots, x_{r-1}]$  st. fix  $p \in \text{Spec } A$ .

①  $B \rightarrow A$  is smooth at  $p$ .

②  $B \rightarrow A/tA$  is finite.

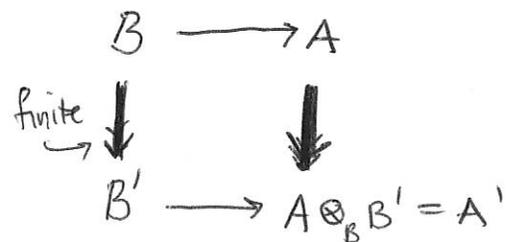


RMK: If  $k = \bar{k}$ , then we can consider all projections  $\text{Spec } A \rightarrow A^{r-1}_k$ .

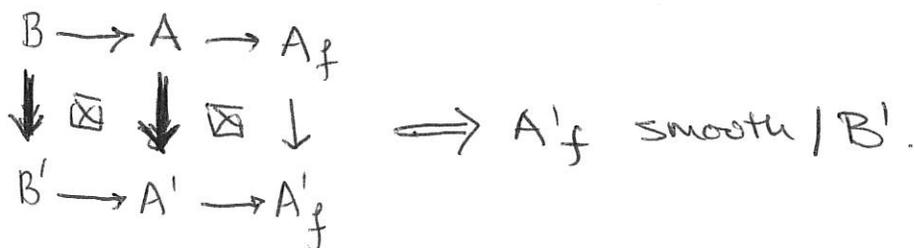
Bertini  $\Rightarrow$  generic projection smooth at  $p$ .

NNL  $\Rightarrow$  generic projection finite along  $\text{Spec}(A/tA)$ .

Proof of  $\otimes$ : Put  $B' = A/tA$ .



Since the smooth locus of  $A/B$  contains  $p$  and is open there is  $f \in A$  st.  $A_f$  is  $B$ -smooth



Finitely many primes  $q_1, \dots, q_n \in \text{Spec } A'_f$  lying over  $p$ , all contained in  $\text{Spec } A'_f \Rightarrow B' \rightarrow A'$  smooth at each  $q_i$ .

Now,  $B' \rightarrow A'$  retracts onto  $B'$  as  $A' = B' \otimes_B A \rightarrow B'$   
 $a_1 \otimes a_2 \mapsto a_1 a_2$

FACT: A section of a smooth morphism is ~~extending~~ a local complete intersection. So  $B'_{f_i} \rightarrow A'_{f_i} \rightarrow B'_{f_i}$  means that  $\text{Ker}(A'_{f_i} \rightarrow B'_{f_i})$  is ~~also~~ defined by a reg.  $^{\text{id}}$  sequence of length 1 (for dimensional reasons).

Put  $S = \cup f_i \subset A'$ .  $S^{-1}A'$  semilocal. and

$S^{-1} \text{Ker}(A' \rightarrow B')$  is locally free of rank 1. over  $S^{-1}A' = A'_g$

and hence must be trivial, ~~trivial~~

As  $\text{Ker}(A' \rightarrow B')$  is finite over  $A$ , there is a  $g \in A$  such that

$$\text{Ker}(A' \rightarrow B') \otimes_A A_g \simeq A'_g$$

Get  $0 \rightarrow A'_g \rightarrow A'_g \rightarrow B'_g \rightarrow 0$  sequence in f.g.  $A$ -mod.

Replace  $g$  by  $gf$  to ensure that  $A'_g / B'$  is smooth.

Given a module  $M \in \mathcal{M}^p(\mathbb{A}/t\mathbb{A}) = \mathcal{M}^p(B')$ . Then  $M \mapsto M \otimes_{B'} A'_g$  is exact.

and defines  $F: \mathcal{M}^p(B') \rightarrow \mathcal{M}^p(A_g)$  as  $A'_g / A_g$  finite.  
 a function.

OTW,  $\text{Tw}_1^{B'}(M, B_g) = 0$ , so  $M_g$

$$0 \rightarrow A'_g \otimes_{B'} M \rightarrow A'_g \otimes_{B'} M \rightarrow B'_g \otimes_{B'} M \rightarrow 0$$

$$\parallel \qquad \parallel$$

$$F(M) \qquad F(M).$$

If  $j: \mathcal{M}^p(B') \rightarrow \mathcal{M}^p(A_g)$   $M \mapsto M_g$  we have an exact sequence of functors  $0 \rightarrow F \rightarrow F \rightarrow j \rightarrow 0$  ~~arises~~ from  $\mathcal{M}^p(B')$  to  $\mathcal{M}^p(A_g)$ .

$$K_* F = K_* F + K_* j \Rightarrow K_* j = 0.$$