Recall: If X noetherian, we defined the Gersten complex w/ X^i = pts of codim i

$$g_n^{\cdot}(X) = \left\{ 0 \to \bigoplus_{x \in X^0} K_n(k(x)) \to \bigoplus_{x \in X^1} K_{n-1}(k(x)) \to \cdots \to \bigoplus_{x \in X^n} K_0(k(x)) \to 0 \right\}.$$

Conjecture: if X regular local then $H^*(g_n(X))$ conc. in degree 0 w/ $H^0(g_n(X)) = K_n(X)$.

Last time: we proved this conjecture holds if $X = \operatorname{Spec} A$ for A essentially smooth / k a field.

Main Idea: We showed that $\mathcal{M}^{p+1}(A) \to \mathcal{M}^p(A)$ induces 0 after applying K_*.

[Here $\mathcal{M}^p(A) = \{$fg modules supported in codim $p\}$] By Popescu's theorem, if A is equicharacteristic and regular then the prime field $\mathbb{F}$ $(= \mathbb{F}_p$ or $\mathbb{Q})$ gives a regular map $\mathbb{F} \to A$ and hence $A = \varinjlim_\alpha B_\alpha$ where each (B_α, m_α) is essentially smooth / $\mathbb{F}$.

By the geometric case of Gersten conj, $\mathcal{M}^{p+1}(B_\alpha) \to \mathcal{M}^p(B_\alpha)$ induces 0 on K-theory for all α, p. But since $B_\alpha \to B_{\alpha'}$ need not be flat, $- \otimes_{B_\alpha} B_{\alpha'}$ does not preserve the codimensional filtration on $\mathcal{M}(B_\alpha)$ — i.e no map $\mathcal{M}^p(B_\alpha) \to \mathcal{M}^p(B_{\alpha'})$.
Need invariants that commute with direct limits!

Recall: I. Localization. For A a ring, $f \in A$, we can consider the Serre subcategory $\mathcal{V}_f \hookrightarrow \mathcal{M}(A)$ consisting of all modules killed by some powers of f. The quotient category $\mathcal{M}(A)/\mathcal{V}_f \simeq \mathcal{M}(A_f)$.
Since K_* doesn't see infinitesimal thickenings (devissage) $K_*(\mathcal{V}_f) = K_*(\mathcal{M}(A/fA))$
$= G_*(A/fA)$.

"exact sequence of categories"

$$\mathcal{V}_f \xrightarrow{i} \mathcal{M}(A) \xrightarrow{\pi} \frac{\mathcal{M}(A)}{\mathcal{V}_f} \quad \text{gives LES}$$

$$\cdots \to K_p(\mathcal{V}_f) \to K_p(\mathcal{M}(A)) \to K_p\left(\frac{\mathcal{M}(A)}{\mathcal{V}_f}\right) \xrightarrow{\partial} K_{p-1}(\mathcal{V}_f) \to \cdots$$

$$\cdots \to G_p(A/fA) \to G_p(A) \to G_p(A_f) \xrightarrow{\partial} G_{p-1}(A/fA) \to \cdots$$

II. Sheaf Cohomology: When we proved Bloch's formula, we had

$$\underline{\underline{g}}_n^{\cdot, X} \left\{ 0 \to \bigoplus_{x \in X^0} (i_x)_*\left(\underline{\underline{K}}_n^{k(x)}\right) \to \bigoplus_{x \in X^1} (i_x)_*\left(\underline{\underline{K}}_{n-1}^{k(x)}\right) \to \cdots \right\}$$

where $i_x : \operatorname{Spec} k(x) \to X$ and $\underline{\underline{K}}_n^Y$ is the Zariski sheaf associated to $U \mapsto K_n(U)$ on the space Y.

If all local rings of X satisfy Gersten's conjecture, then the above is a flabby resolution of $\underline{\underline{K}}_n^X$ so $H^q(X, \underline{\underline{K}}_n^X) = H^q(\Gamma(X, \underline{\underline{g}}^{\cdot, X})) = H^q(g_n^{\cdot}(X))$.

Note that the LHS commutes with direct limits.

III. $X = \operatorname{Spec} A \quad X_f = \operatorname{Spec}(A_f) \quad Z = \operatorname{Spec}(A/fA)$. f nonzero divisor.

$g_\cdot(X)$, $g^\cdot(Z)$ and $g^\cdot(X_f)$ fit into exact sequence: Fix p.

$$0 \to \underbrace{\bigoplus_{\boxed{\substack{x\in Z \\ x\in X^p}}} K_q(k(x))}_{\substack{g^{p-1}_{q-1}(Z) \\ \| \\ (g_{q-1}(Z)[-1])^p}} \to \underbrace{\bigoplus_{x\in X^p} K_q(k(x))}_{g^p_q(X)} \twoheadrightarrow \underbrace{\bigoplus_{x\in (X_f)^p} K_q(k(x))}_{g^p_{q}(X_f)} \to 0$$

aka $x \in Z^{p-1}$

Get $\quad 0 \to g^\cdot_{n-1}(Z)[-1] \to g^\cdot_n(X) \to g^\cdot_n(X_f) \to 0.$

Key Lemma:

~~Thm~~: Let (A, m) reg. loc. equicharacteristic. Let $f \in m - m^2$. Then

$$H^p_{Zar}(X_f, \underline{K}_n) = \begin{cases} K_n(X_f) & p = 0 \\ 0 & p > 0. \end{cases}$$

Pf: (Step 1: Suppose A essentially smooth / $\mathbb{F}$). $\quad$ Z regular.

$0 \to g^\cdot_{n-1}(Z)[-1] \to g^\cdot_n(X) \to g^\cdot_n(X_f) \to 0$ induces for $p \geq 1$

$$\begin{tikzcd}
H^p(g^\cdot_n(X)) \arrow[r] \arrow[d, equal] & H^p(g^\cdot_n(X_f)) \arrow[r] \arrow[d, equal] & H^{p+1}(g_{n-1}(Z)[-1]) \arrow[d, equal] \\
0 & H^p_{zar}(X_f, \underline{K}_n) & H^p(g_{n-1}(Z)) \arrow[d, equal] \\
 & & 0
\end{tikzcd}$$

(X satisfies Gersten conj) $\quad$ every local ring of X_f satisfies Gersten conj.

For $p = 0$, we have

$= H^1(g_{n-1}(Z)[-1])$.

$$\begin{tikzcd}
 & 0 \arrow[r] & H^0(X, \underline{K}_n) \arrow[r] & H^0(X_f, \underline{K}_n) \arrow[r] & H^0(Z, \underline{K}_{n-1}) \arrow[r] & 0 = H^1(g_n(X)) \\
K_n(Z) \arrow[rr, "0"] & & K_n(X) \arrow[r] \arrow[u, "c_1", "\cong"'] & K_n(X_f) \arrow[r, "\partial"] \arrow[u, "c_2"] & K_{n-1}(Z) \arrow[r, "0"] \arrow[u, "c_3", "\cong"'] & K_{n-1}(X)
\end{tikzcd}$$

] ~~Eternal~~ sections of sheaf

] sections of presheaf.

by gersten. $\quad$ X, Z local so flanking arrows c_1, c_3 isos $\quad$ by gersten.

c_2 iso by 5-lemma.

Step 2: Pass to the limit

$A = \varinjlim_\alpha B_\alpha$ where (B_α, m_α) ess smooth over prime field $\mathbb{F}$.

~~Choose~~ $f \in m - m^2$ comes from some $f_\alpha \in B_\alpha$. Since $B_\alpha \to A$ local, $f_\alpha \in m_\alpha - m_\alpha^2$. So $A_f = \varinjlim_{\alpha' \geq \alpha} (B_{\alpha'})_{f_{\alpha'}}$. By notational abuse $A_f = \varinjlim_{\alpha} (B_\alpha)_{f_\alpha}$

$X_\alpha = \operatorname{Spec} B_\alpha \quad X_{f_\alpha} = \operatorname{Spec} (B_\alpha)_{f_\alpha}. \quad X_f = \varprojlim_\alpha (X_\alpha)_{f_\alpha}.$

Note that K_n commutes w/ inverse limits of affine morphisms. ie

$K_n(X_f) = \varinjlim K_n((X_\alpha)_{f_\alpha})$.

GROTHENDIECK LIMIT THEOREM: Let Y_α be a ~~family~~ projective system of affine schemes (noetherian) $Y = \varprojlim_\alpha Y_\alpha$. (w/ Y noetherian). Let $\mathcal{F}_\alpha$ be a compatible system of abelian gp valued sheaves on the Y_α. Then $\varinjlim_\alpha H^q(Y_\alpha, \mathcal{F}_\alpha) = H^q(Y, \mathcal{F})$. where $\mathcal{F} = \varinjlim_\alpha \mathcal{F}_\alpha$

Thus, $\underline{\underline{K}}_n$ give compatible system of sheaves on $(X_\alpha)_{f_\alpha}$. we can apply the limit theorem. □

Key Lemma 2: Let $X = \operatorname{Spec} A$ w/ A reg local equichar. Then. $f \in m - m^2$.
$Z = A/fA$.

Then the natural map $G_n(Z) \to G_n(X)$ is 0. [Hence $K_n(Z) \to K_n(X)$ is 0]
Since Z, X regular.

Pf: If A ess. smooth / $\mathbb{F}$ then ∃ factorization

$$\begin{tikzcd} G_n(Z) \arrow[r] \arrow[dr] & G_n(X) = K_n(\mathcal{M}^0(X)) \\ & K_n(\mathcal{M}^1(X)) \arrow[u, "0"'] \end{tikzcd}$$

$0 \leftarrow$ By Gersten.

Since X, X_f, Z regular, we can replace G's by K's in the localization sequence:

$K_n(Z) \to K_n(X) \to K_n(X_f) \xrightarrow{\partial} K_{n-1}(Z)$. Saying $K_n(X) \to K_n(X_f)$ is 0 is equivalent ~~suffices~~ to showing that $K_n(X) \to K_n(X_f)$ injective. But as before, $A_f = \varinjlim_\alpha (B_\alpha)_{f_\alpha}$.

and $K_n(A) \to K_n(A_f)$ is the limit of maps $K_n(B_\alpha) \to K_n((B_\alpha)_f)$ which are injective since $B_\alpha / \mathbb{F}$ is ess. smooth.

<u>Proof of main theorem</u>: We have $X = \operatorname{Spec} A$ $f \in m - m^2$ A local equichar regular... $Z = \operatorname{Spec}(A/fA)$.

<u>WTS</u>: $H^q(g_n^\cdot(X)) = \begin{cases} K_n(X) & \text{if } q = 0 \\ 0 & q > 0. \end{cases}$

Consider $0 \to g_{n-1}^\cdot(Z)[-1] \to g_n^\cdot(X) \to g_n^\cdot(X_f) \to 0$.

Assume by induction that every equicharacteristic reg local ring of dim less than dim A satisfies Gersten's conjecture.

Hence all local rings of X_f and satisfy Gersten's conjecture.

So $H^q(X_f, \underline{K}_n) = H^q(g_n^\cdot(X_f))$ and $H^q(g_n^\cdot(Z)) = 0 \ \forall q > 0$.

if $q \geq 2$, we get

$$\begin{tikzcd}
H^q(g_{n-1}^\cdot(Z)[-1]) \arrow[r] \arrow[d, equal] & H^q(g_n^\cdot(X)) \arrow[r] & H^q(g_n^\cdot(X_f)) \arrow[d, equal] \\
H^{q-1}(g_{n-1}(Z)) \arrow[d, equal] & & H^q(X_f, \underline{K}_n) \arrow[d, equal] \\
0 & & 0 \text{ (by Lemma 1)}.
\end{tikzcd}$$

In low degree, we have.

$$0 \to H^0(g_n(X)) \to H^0(g_n(X_f)) \to H^1(g_{n-1}(Z)[-1]) \to H^1(g(X)) \to H^1(g^\cdot(X_f)) = 0.$$

$\| $

$H^0(g_{n-1}(Z)[-1])$

$$\begin{tikzcd}
0 \arrow[r] & H^0(g_n(X)) \arrow[r] & H^0(X_f, \underline{K}_n) \arrow[r] & H^0(Z, \underline{K}_{n-1}) \arrow[r] & H^1(g(X)) \arrow[r] & 0 \\
K_n(Z) \arrow[r, "0"] & K_n(X) \arrow[r] \arrow[u, "?"] & K_n(X_f) \arrow[r] \arrow[u, "\cong"] & K_{n-1}(Z) \arrow[r, "0"] \arrow[u, "\cong"] & K_{n-1}(X) \arrow[r] & {}
\end{tikzcd}$$

Localization + Lemma 2.

$H^1(g_n(X)) = 0$

$H^0(g_n(X)) \cong K_n(X)$ by 5-lemma.