Néron-Popescu seminar.

Tucker.

25 September 2015.

Swan paper.

Recall. R-algebra A. (All are commutative.)

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, dashed] \\ B \arrow[r] & B/I \end{tikzcd}$$

$$0 \to I \to B \to B/I \to 0$$

$I^2 = 0$

A is quasi-smooth (resp. quasi-étale) if a lift (resp. a unique lift) exists.

Smooth (étale): just add f.p.

Add essentially for ess. f.p.

Low dimensional cohomology.

$$R[x_i]/I = A.$$

$$I/I^2 \xrightarrow{d} \bigoplus_{i \in I} A \cdot dx_i \cong \Omega_{R[x]/R} \otimes_{R[x]} A.$$

Rem. d is an A-module map.

$$f \longmapsto \sum \frac{\partial f}{\partial x_i} dx_i$$

$$\operatorname{coker}(d) = \Omega^1_{A/R},$$

$$\ker(d) = \Gamma_{A/R}.$$

Def. Tensor with M. Get

$\operatorname{coker}(d \otimes M) := H_0(R,A,M)$, André-Quillen homology.

$\ker(d \otimes M) := H_1(R,A,M)$.

Thm (3.4 from Swan). TFAE.

(1) $R \to A$ is qsm.

(2) d is a split mono.

$\Updownarrow$ easy

(3) $\Gamma_{A/R} = 0$ and $\Omega^1_{A/R}$ is projective.

Thm. TFAE.

(4) d is an iso.

Easy.

(5) $\Gamma_{A/R} = 0$ and $\Omega^1_{A/R} = 0$.

(6) qsé.

The smooth locus.

$R \to A$ f.p.

Fix $R[\underline{x_i}]/I \xrightarrow{\simeq} A$.

Take any finite set $f_1, \dots, f_l$ of I.

Take $\left[\frac{\partial f_i}{\partial x_j}\right]$, matrix of partials.

$\Delta(f_1, \dots, f_l)$ = dets of maximal minors.

This is inside $R[\underline{x}]$ or in A.

Swan says inside A.

Def. $H_{A/R}$ = radical of $\sum_{f_1,\dots,f_l \in I} \Delta(f_1,\dots,f_l) \cdot [\langle f_1,\dots,f_l\rangle : I]$.

$$[\langle f_1,\dots,f_l\rangle : I] = \{f \in R \mid fI \subseteq \langle f_1,\dots,f_l\rangle\} \supseteq I.$$

Thm (4.1). A f.p. R-algebra, $P \in \operatorname{Spec} A$.

A_P is ess. smooth over R iff $P \not\supseteq H_{A/R}$.

Q. Why isn't $H_{A/R}$ as defined the zero ideal of A? Because typo. Corrected now.

Functoriality.

$$\begin{tikzcd}
 & R[x] \arrow[dr] \arrow[dd, dashed, "\text{choose a lift}"] & \\
R \arrow[ur] \arrow[rr] \arrow[d] & & R[x]/I \cong A \arrow[d] \\
S \arrow[rr] \arrow[dr] & {} & S[y]/J \cong B \\
 & S[y] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I/I^2 \arrow[r, "d"] \arrow[d] & \bigoplus A\,dx_i \arrow[d] \\
J/J^2 \arrow[r, "d"] & \bigoplus B\,dy_i
\end{tikzcd}$$

This quickly can be used to show that Ω and Γ are independent of presentation.

Ex. If $A = R/I$, $\Omega_{A/R} = 0$,

$\Gamma_{A/R} \cong I/I^2$.

Lemma. Commutes with filtered colimits.

Jacobi-Zariski sequence. $R \to A \to B$. Get natural maps

$$\underbrace{B \otimes_A \Gamma_{A/R}}_{} \to \Gamma_{B/R} \to \Gamma_{B/A} \to B \otimes_A \Omega_{A/R} \to \Omega_{B/R} \to \Omega_{B/A} \to 0 .$$

if $\Omega_{A/R}$ is
flat as an A-module.

$A = R[x]/I$, $B = R[y]/J = R[x, y]/L = \langle x, J \rangle$ Get the following.

$$\begin{tikzcd}
B \otimes_A I/I^2 \arrow[r] \arrow[d] & L/L^2 \arrow[r, "x \mapsto 0"] \arrow[d] & J/J^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus B dx_i \arrow[r] & (\oplus B dx_i) \oplus (\oplus B dy_i) \arrow[r, "dx_i \mapsto 0"'] & \oplus B dy_i \arrow[r] & 0
\end{tikzcd}$$

See why we need flatness. Snake lemma is OK.

Added 9/26.

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R[x] \arrow[r] \arrow[d, "\phi_i"] & A = R[x]/I \arrow[d] \\
S \arrow[r] & S[y] \arrow[r] & B = S[y]/J
\end{tikzcd}
\qquad
\begin{tikzcd}
I/I^2 \arrow[r] \arrow[d, "\phi_i"'] & \oplus A dx_i \arrow[d, "d\phi_i"] \arrow[dl, dashed, "dx_i \mapsto h(x_i)"'] \\
J/J^2 \arrow[r] & \oplus B dy_j
\end{tikzcd}$$

Need to show that induced maps of complexes are independent of choice of ϕ_i. If ϕ_0, ϕ_1 are two choices,
$h = \phi_0 - \phi_1 : R[x] \to J$.

Get, $h(f x_i) = f h(x_i) \in J^2$ if $f \in I$, so get a map
$\oplus A dx_i \to J/J^2$.

Get a chain homotopy.

$\phi_0(f) - \phi_1(f)$

$$df = \sum \bar{a}_i dx_i \longrightarrow \sum a_i h(x_i)$$

OK. Looks good.

Functoriality.

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R[x] \arrow[r] \arrow[d, shift right, "\phi_0"'] \arrow[d, shift left, "\phi_1"] & A \cong R[x]/I \arrow[d, "F"] \\
S \arrow[r] & S[y] \arrow[r] & B \cong S[y]/J
\end{tikzcd}$$

$$\begin{tikzcd}
I/I^2 \arrow[r, "d_A"] \arrow[d, shift right, "\phi_0"'] \arrow[d, shift left, "\phi_1"] & \oplus A\,dx_i \arrow[dl, "h"'] \arrow[d, shift right, "d\phi_0"'] \arrow[d, shift left, "d\phi_1"] \\
J/J^2 \arrow[r, "d_B"'] & \oplus B\,dy_j
\end{tikzcd}$$

$$d\phi_0(dx_i) = \sum \frac{\partial \phi_0(x_i)}{\partial y_j}\,dy_j.$$

$$h(dx_i) = \phi_0(x_i) - \phi_1(x_i) \mod J^2.$$

$\underline{d_B \circ h = d\phi_0 - d\phi_1.}$

$$\begin{aligned}
d_B(h(dx_i)) &= d_B(\phi_0(x_i)) \\
&\quad - d_B(\phi_1(x_i)) \\
&= \sum \frac{\partial \phi_0(x_i)}{\partial y_j}\,dy_j \\
&\quad - \sum \frac{\partial \phi_1(x_i)}{\partial y_j}\,dy_j \\
&= d\phi_0(x_i) - d\phi_1(x_i).
\end{aligned}$$

$\underline{h \circ d_A = \phi_0 - \phi_1.}$

Let $p(x) = p \in I/I^2$. Then,

$$\phi_0(p(x)) = p(\phi_0(x)) = p(\phi_0(x_i) \mid i \in I).$$

Now,

$$\begin{aligned}
\phi_0(p(x)) - \phi_1(p(x)) &= p(\phi_0(x_i) \mid i \in I) - p(\phi_1(x_i) \mid i \in I) \\
&= p(\phi_0(x_i) \mid i \in I) - p(\phi_0(x_i) + q(x_i) \mid i \in I)
\end{aligned}$$

Looking at the expansion of $p(\phi_0(x_i) + q(x_i))$.

Warning. $q(x_i) = -(\phi_0(x_i) - \phi_1(x_i))$

$$= p(\phi_0(x_i)) - \left(p(\phi_0(x_i)) + \sum_{i \in I} \frac{\partial p}{\partial x_i}(\phi_0(x))\, q(x_i) + J^2 \right)$$

Thanks Kevin!

$$\equiv -\sum \frac{\partial p}{\partial x_i}(\phi_0(x))\, q(x_i)$$

$$\parallel$$

$$\sum \frac{\partial p}{\partial x_i}(\phi_0(x_i))(\phi_0(x_i) - \phi_1(x_i)).$$

Proof of 3.4. Assume (2).

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A = R[x]/I \arrow[d] & \\ B \arrow[r] & B/J \arrow[r] & 0 \end{tikzcd}$$

$J^2 = 0$

Choose a lift of $R[x] \to B/J$ to $R[x] \xrightarrow{\psi} B$.

$x_i \mapsto \bar{b}_i \qquad x_i \mapsto b_i$.

Adjust b_i by $\delta_i \in J$. If $f \in R[x]$,

$$f(b_i + \delta_i) = f(b_i) + \sum \frac{\partial f}{\partial x_i}(b_i)\,\delta_i$$

since $J^2 = 0$.

Want to choose δ_i so that $R[x] \to B$ factors through A.

$$f \longmapsto \frac{\partial f}{\partial x_i}$$

Can we get:

$$\begin{tikzcd} I/I^2 \arrow[r] \arrow[d, "\psi"'] & \bigoplus A\,dx_i \arrow[d, "-\delta"] & dx_i \arrow[d, mapsto] \\ J/J^2 \arrow[r, "\cong"] & J & \delta_i \end{tikzcd}$$

?

A-module structure on J via $I/I^2 \to J/J^2 = J$.

Yes, of course if $I/I^2 \to \bigoplus A\,dx_i$ is split injective.