10/2/15

Recall: $R \to A$. $A = R[\underline{x}]/I$. presentation.

$$I/I^2 \xrightarrow{d} \bigoplus A\,dx_i$$

$$f \longmapsto \sum \frac{\partial f}{\partial x_i} dx_i$$

$\ker d = \Gamma_{A/R}$ $\quad$ $\operatorname{coker} d = \Omega_{A/R}$

Thm: TFAE

1) A is quasi-smooth $/R$ (0-smooth)
2) d split injective
3) $\Gamma_{A/R} = 0$, $\Omega_{A/R}$ projective.

Thm: TFAE

(A) A is quasi-étale $/R$ (0-étale)
(B) d isomorphism
(C) $\Gamma_{A/R}, \Omega_{A/R} = 0$.

Proof of (2) ⇒ (1)

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, "\varphi"'] & R[\underline{x}]/I \arrow[l, two heads] \arrow[dl, "\bar{\varphi}"] \\ B \arrow[r] & B/J & J^2 = 0. \end{tikzcd}$$

$\varphi(f) = f(\underline{b})$

Say $\varphi(x_i) = b_i$ where $\bar{\varphi}(x_i) = \bar{b}_i \in B/J$.

Want to choose $\delta_i \in J$ so that

$f(\underline{b} + \underline{\delta}) = 0 \quad \forall f \in I$.

"<u>Newton's Method</u>"

$$f(\underline{b}+\underline{\delta}) = f(\underline{b}) + \sum \frac{\partial f}{\partial x_i}(\underline{b})\,\delta_i$$

From the diagram $\varphi(I) \subset J$.

Get an induced $I/I^2 \to J/J^2 = J$.

Note that the existence of the $\delta_i \in J$ is equivalent to the existence of a lift

$$\begin{tikzcd}
I/I^2 \arrow[r, "d"] \arrow[d, "\varphi"'] & \bigoplus A\,dx_i \arrow[dl, dashed] \\
J &
\end{tikzcd}$$

If d is split injective, then we in fact get a $\psi : \bigoplus A\,dx_i \to I/I^2$ st

$\psi \circ d = \mathrm{Id}$. So the lift is $\varphi\psi$. //

$(1) \Rightarrow (2)$. Let $B = R[X]/I^2$ $\quad J = J/J^2$.

~~Get the lift~~

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A \arrow[d, "\varphi"] \arrow[dl, dashed] & R[X] \arrow[l] \\
B \arrow[r] & B/J &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
I/I^2 \arrow[r, "d"] \arrow[d, "\varphi", "\simeq"'] & \bigoplus A\,dx_i \arrow[dl, dashed] \\
J/J^2 &
\end{tikzcd}$$

Lift here exists by quasi-smoothness

Implies existence of lift here.

But map induced by φ is an iso, so the dotted diagonal arrow gives the splitting.

Theorem in the quasi-étale setting is proved similarly.

<u>Jacobi-Zariski Sequence</u>: $R \to A \to B$.

$$\Gamma_{B/R} \to \Gamma_{B/A} \to \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to \Omega_{B/A} \to 0$$

if $\Omega_{A/R}$ is A-flat get an additional term $\Gamma_{A/R} \otimes B$ on left.

<u>RMK</u>: If $B = A/I$ then $\Omega_{B/A} = 0$. and $\Gamma_{B/A} = I/I^2$ so we recover the "2nd fundamental sequence"

$$\otimes \quad I/I^2 \xrightarrow{d} \Omega_{A/R} \otimes B \longrightarrow \Omega_{B/R} \to 0.$$

cf. Matsumura

If $B = A/I$ and $R \to A$ is quasi-smooth then we can use the above $\otimes$ to compute $\Omega_{B/R}, \Gamma_{B/R}$.

<u>Localization</u>: $S \subset A$ multiplicative. $R \to A$.

Then
$$S^{-1}\Omega_{A/R} = \Omega_{S^{-1}A/R}$$
$$S^{-1}\Gamma_{A/R} = \Gamma_{S^{-1}A/R}.$$

Use JZ + fact that $A \to S^{-1}A$ is quasi-étale.

RMK 2:

multiplicative

$A = R[\underline{x}]/I \supset S$ lift S to T in $R[\underline{x}]$

Use $T^{-1}R[\underline{x}] \to S^{-1}A$. to compute...

and get Cor: $S^{-1}A$ quasi-smooth $/R$ $\Leftrightarrow$

$$S^{-1}(I/I^2) \xrightarrow{d} \bigoplus S^{-1}A\,dx_i$$

is split injective.

Cor: A finite presented $/R$. $S \subset A$ multiplicative $S^{-1}A$ essentially smooth $/R$. Then $\exists\, s \in S$ st. $A[1/s]$ smooth $/R$.

Pf: Know $S^{-1}(I/I^2) \xrightarrow{d} S^{-1}A\,dx_i$ split injective. Finite presentation hypothesis $\Rightarrow$ ~~both modules are finite / A~~ ~~$S^{-1}A$~~ I/I^2, $\bigoplus A\,dx_i$ fg $/A$. Now find a single $s \in S$ via the usual trick.

Cor: $R \to A \to \Lambda$ and

$\uparrow$ finite pres. (at A)

$R \to A \to B \to \Lambda$ Then there

$\uparrow$ ess. sm $/R$. (at B)

exists $R \to A \to C \to \Lambda$

$\uparrow$ smooth$/R$. (at C)

B ess / smooth over $R \Rightarrow$

Pf: $R \to C \to S^{-1}C = B \to \Lambda$.
$\uparrow$ finite pres.

$R \to A \to B$ By finite pres. hyp.

$A = R[y_1 \cdots y_n]/(f_1 \cdots f_r)$.

$y_i \mapsto c_i/t \in B$ $t \in S$ $c_i \in C$.

Can find $R \to C[\frac{1}{s}]$ smooth for some $s \in S$.

$R \to R[y_1 \cdots y_n]/(f_i) \to C[\frac{1}{stu}] \to \Lambda$

$y_i \longmapsto c_i/t$

may need to invert some u to kill $\langle f_i \rangle$

$R \to A$ $A = R[\underline{x}]/I$ finite presentation.

$r \times r$ minors of Jacobian matrix

$H_{A|R}$ = radical in $\underline{\underline{A}}$ of $\sum_{f_1 \cdots f_n \in I} \Delta_r(f_1, \cdots f_r) \cdot [(f_1 \cdots f_r) : I]$

Thm 4.1: A_p / R ess sm. $\Leftrightarrow H_{A|R} \not\subset P$.
$\forall p \in \operatorname{spec} A$.

Pf: Sketch: $H_{A|R} \not\subset P \Rightarrow \exists f_1 \cdots f_r \in I$ st.

$\Delta_r(f_1 \cdots f_r) \cdot [(f_1 \cdots f_r) : I] \not\subset P$.

1) Hence, some $r \times r$ minor ~~of~~ not in $\mathfrak{p}$.
say the first one.

2) $[(f_1 \cdots f_r) : I] \not\subset P \Rightarrow$

$$\operatorname{ann}\left(I/(f_1 \cdots f_r) \right) \not\subset P.$$

so $(f_1 \cdots f_r)_p = I_p$.

$$\begin{tikzcd}
A_p^{\oplus r} \arrow[r, two heads, "\psi"] \arrow[drr, bend right=15, "\cong"'] & (I/I^2)_p \arrow[r, "d_p"] & \oplus A_p \, dx_i \arrow[d, "\text{projection onto first } r \text{ factors}"] \\
& & A_p^{\oplus r}
\end{tikzcd}$$

$e_i \longmapsto f_i$

determined by first $r \times r$ submatrix of $\left(\frac{\partial f_i}{\partial x_j}\right)$.

By assumption this has unit determinant.

Thus ψ is injective ~~as well~~. (hence an iso.)

d_p is therefore injective:

$$0 \to (I/I^2)_p \xrightarrow{d_p} \oplus A_p \, dx_i \to A_p^{\oplus r} \to 0.$$

Sequence splits b/c $A_p^{\oplus r}$ projective. //
Now use criterion for quasi-smoothness..