Smoothness and Field Extensions

Thm (5.1) (Cartier-MacLane) E/F finitely gen.

Then $\dim \Omega_{E/F} = \dim_E \Gamma_{E/F} + \operatorname{tr\,deg}(E/F)$

Pf: Choose $x_1 \cdots x_n$ transcendence ~~tr~~ basis for E/F.
Then $F(x_1 \cdots x_n) \to E$ is finite.

$E = F(x_1 \cdots x_n)[y_1 \cdots y_m] / (f_1 \cdots f_n)$. image of y_i in E

f_i minimal polynomial of the monogenic ext'n. $F(x_1 \cdots x_n)[y_1 \cdots y_{i-1}] / (f_1 \cdots f_{i-1}) \, [y_i] \hookrightarrow E$

Thus, ~~[illegible]~~ $(f_1 \cdots f_m)$ form a reg seq.

$$\underbrace{(f_1 \cdots f_m)}_{= I}$$

$E = A/I$ where $A = F(x_1 \cdots x_n)[y_1 \cdots y_m]$.

Look at JZ sequence: since $F \to A$ is 0-smooth.

$$0 \to \Gamma_{E/F} \to \underbrace{I/I^2}_{\text{free of dim } m} \to \underbrace{\Omega_{A/F} \otimes E}_{\text{has dim } m+n} \to \Omega_{E/F} \to 0$$

Now take alternating sum across ~~@~~ sequence. //

<u>Prop 3.5</u> $R \to A$ 0-smooth. $B = A/I$.
Then $\Gamma_{A/R} = 0$ $\Omega_{A/R}$ projective.

$\Gamma_{B/A} = I/I^2$ JZ becomes

$$0 \to \Gamma_{B/R} \to I/I^2 \xrightarrow{d} B \otimes_A \Omega_{A/R} \to \Omega_{B/R} \to 0.$$

<u>Cor 5.2</u>: $F \subset E \subset K$ field ext'ns. Then

$$0 \to K \otimes_E \Gamma_{E/F} \to \Gamma_{K/F} \to \Gamma_{K/E} \to \Omega_{E/F} \otimes K$$
$$\to \Omega_{K/F} \to \Omega_{K/E} \to 0.$$

<u>Pf</u>: <u>Step 1</u>: Reduce to finitely gen. ext'ns.
(Easy b/c JZ commutes w/ direct limits)

<u>Step 2</u>: For $R \to A \to B$ we have

$$B \otimes_A \Gamma_{A/R} \to \Gamma_{B/R} \to \Gamma_{B/A} \to B \otimes \Omega_{A/R} \to \Omega_{B/R} \to \Omega_{B/R} \to 0$$

when $\Omega_{A/R}$ is flat.

Now specify to $F \to E \to K$. (Flatness condition trivially satisfied).

We have

$$0 \to T \to K \otimes_E \Gamma_{E/F} \to \Gamma_{K/F} \to \Gamma_{K/E} \to \Omega_{E/F} \otimes K$$
$$\to \Omega_{K/F} \to \Omega_{K/E} \to 0.$$

kernel

Looking at alternating sum and using 5.1, $\dim_K E = 0$

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Def: A field ext'n E/F is called separable if $F \to E$ is quasi-smooth.

(The usual def. is E/F separable $\iff$ $E \otimes_F L$ reduced $\forall L/F$).

~~Once we have the equivalence of definitions, we get~~

1. $F \subset E \subset K$ $\quad K/F$ sep $\iff$ E/F and K/E separable.

2. If E/F is algebraic let $a \in E$ and consider $F(a) = E'$.

From 5.1 $\Omega_{E'/F} = 0 \iff \Gamma_{E'/F} = 0$.

Let f be the minimal poly for a.

$$I = (f(x)) \subset F[x]$$

$$\begin{tikzcd}
\Gamma_{E'/F} \arrow[r] & I/I^2 \arrow[r, "d"] \arrow[d, "\wr"] & E'dx \arrow[r] \arrow[d, "\wr"] & \Omega_{E'/F} \\
& E \arrow[r, "f'(a)"] & E &
\end{tikzcd}$$

so $f'(a) = 0 \iff \Gamma_{E'/F} \neq 0$. //

Cor 5.3 E/F ess. finite type (ie fg field ext)

Then $\Omega_{E/F} = 0 \iff E/F$ finite sep.

Moreover, if A is a local ess. finite F alg then

A ess ét $/F \iff A$ finite sep field ext of F.

~~(Pf for local ring case):~~

~~$E = A/m$~~

~~Assume A/F ess ét. Then $\Omega_{A/F}$~~

A Bad Extension: char $F = p$. x_1 transcendtl over F

$E = F(x_1, x_2, \cdots)$

$x_i^p = x_{i-1}^p$

E/F has tr. deg 1 but $\Gamma_{E/F}$ and $\Omega_{E/F} = 0$.

Def: A field ext'n K/F is separably generated if there is an E

$K \supset E \supset F$

w/ K/E sep alg, E/F purely transcendtl.

With $E = F(x_1 \cdots x_n)$ we call $x_1 \cdots x_n$ a separating transcendence base.

Def: A set of elements $\{x_i\} \subset K$ is a Ω-basis for $K|F$ if $\{dx_i\}$ form a K-basis for $\Omega_{K|F}$.

Cor 5.4: A finitely gen. field ext'n $K|F$ is separable $\iff$ $K|F$ separably generated.

In this case the notions of Ω-basis and sep. trans. basis coincide.

Pf: ($\Leftarrow$) Assume $E = F(x_1 \cdots x_n)$ purely transcendental $K|E$ alg sep.
Look at J-Z for $F \to E \to K$.
tells us
$\Gamma_{K|F} = 0$
$K \otimes_E \Omega_{E|F} \to \Omega_{K|F}$ is an iso.
$\Omega_{E|F}$ is free on dx_i

($\Rightarrow$). Assume $F \to K$ is quasi-smooth.
Choose a ~~basis~~ Ω-basis $\{x_1 \cdots x_n\}$ for $\Omega_{K|F}$. Put $E = F(\{x_i\})$.
$K \otimes \Omega_{E|F} \xrightarrow{\varphi} \Omega_{K|F}$ is surjective.

As $K|F$ is sep. the JZ-seq. says φ is also injective.

Thus the $\{x_i\}$ are alg. independent.

$JZ \Rightarrow \Omega_{K|E} = 0$.

By 5.3, K/E is finite separable.