① (10/16/15 - W. Zhang)

§5 + §6: Separability via Smoothness

**Def** E/F separable iff E g-sm / F.

**Goal** (Maclane's Criterion) E/F separable iff E/F geometrically regular over F.

**Thm 5.1**: E/F field ext ess of fin type (e.g. $E = F(x_1, \dots, x_n)$) $\dim_E \Omega_{E/F} = \dim_E \Gamma_{E/F} + \operatorname{trdeg} E/F$

**Corollary 5.3**: E/F ess fin type

$\Omega_{E/F} = 0 \iff E/F$ finite separable

**Def**: A set $\{x_i\}$ of elements in K is an $\Omega$-basis of K/F if $\{dx_i\}$ is a basis of $\Omega_{K/F}$ over K. (differential basis)

**Prop 5.4** separably generated $\iff$ separable

**Def** (p-basis) char $F = p > 0$, K/F
$a_1, \dots, a_n \in K$ are p-indep / F if

$[K^p(a_1, \dots, a_n) : K^p(F)] = p^n$.

$C \subseteq K$ is <u>$p$-indep</u> if every finite subset of $C$ is $p$-indep.

A subset $C \subseteq K$ is a <u>$p$-base</u> if $C$ is <u>$p$-indep</u> and $K = K^p(F, C)$.

<u>Rmk</u> $C$ is a $p$-basis $\iff$

$a_1^{e_1} \cdots a_n^{e_n}$ $(0 \le e_i < p)$ form a basis for $K$ over $K^pF$ for all distinct $a_1, \dots, a_n \in C$

<u>Lemma 5.6</u>: (char $p > 0$) $K/F$, $K^p \subseteq F \subseteq K$

$\{x_i\} \subseteq K$ set $a_i = x_i^p$

Then ① $K = F(\{x_i\}) \iff \{dx_i\}$ generate $\Omega_{K/F}$

② $\{dx_i\}$ lin indep in $\Omega_{K/F} \iff$

$$\begin{array}{c} F[\{T_i\}]/(T_i^p - a_i) \\ \downarrow \\ K \end{array} \qquad T_i \mapsto x_i \quad \text{is inj.}$$

<u>Corollary 5.7</u>

$\{x_i\}$ is a $p$-basis for $K/F$

$\iff$

$\{x_i\}$ is a $\Omega$-basis for $K/F$

③

Pf: $dK^p = 0$ in $\Omega_{K/F}$

$\Rightarrow \Omega_{K/F} = \Omega_{K/K^pF}$

Now replace $F$ by $K^pF$ and use 5.6. ✓

Thm 6.1 (Faltings) $k$ = field

$(A, \underline{m})$ = geom regular local $k$-alg

$K = A/\underline{m}$ $\quad (k \longrightarrow A \longrightarrow K)$

we have exact sequence (JZ seq)

$$0 \longrightarrow \Gamma_{K/k} \longrightarrow \underline{m}/\underline{m}^2 \xrightarrow{d} K \otimes \Omega_{A/k} \rightarrow \Omega_{K/k} \longrightarrow 0$$

Corollary 6.2 Same hypothesis as 6.1.

Then $\dim_K \Gamma_{K/k} < \infty$

Corollary 6.3 (Maclane's Criterion)

$E/F$ separable $\Leftrightarrow$ ~~it~~ geometrically regular

Pf: ($\Leftarrow$) Assume $E/F$ geometrically regular. Then $6.1 \Rightarrow \Gamma_{E/F} = 0$.

($\Rightarrow$) (well-known)

A smooth local $k$-algebra must be regular.

④

<u>Proof of 6.1</u>: Need $\Gamma_{K/k} \hookrightarrow \mathfrak{m}/\mathfrak{m}^2$

$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad \Gamma_{K/A}$

$\qquad\qquad\qquad\qquad\qquad\qquad A \twoheadrightarrow K$

May assume char $k = p > 0$.
$\mathbb{F} \subseteq k$ is a prime field
$\Rightarrow \mathbb{F}$ perfect $\Rightarrow \Gamma_{K/\mathbb{F}} = 0$.

$$\begin{tikzcd}
\mathbb{F} \arrow[r, hook] \arrow[d] & k \arrow[r] \arrow[d] & K \arrow[d] \\
\mathbb{F} \arrow[r] & A \arrow[r, two heads] & K
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{K/k} \arrow[r] \arrow[d] & K \otimes_k \Omega_{k/\mathbb{F}} \arrow[r] \arrow[d, "⊛"] & \Omega_{K/\mathbb{F}} \arrow[r] \arrow[d] & \Omega_{K/k} \arrow[r] & 0 \\
0 \arrow[r] & \Gamma_{K/A} \arrow[r] & K \otimes_A \Omega_{A/\mathbb{F}} \arrow[r] & \Omega_{K/\mathbb{F}} \arrow[r] & 0 &
\end{tikzcd}$$

reduce to proving

$$K \otimes_k \Omega_{k/\mathbb{F}} \overset{⊛}{\hookrightarrow} K \otimes_A \Omega_{A/\mathbb{F}}.$$

Will show

$$K' \otimes_k \Omega_{k/\mathbb{F}} \hookrightarrow K' \otimes_A \Omega_{A/\mathbb{F}} \qquad K'$$

even bigger, depending on elements of ker ⊛

Assume $x_1, \dots, x_n \in k$ are such that $dx_1, \dots, dx_n$ are lin indep in $\Omega_{k/\mathbb{F}}$.

⑤

Set $k' = k(x_1^{1/p}, \dots, x_n^{1/p})$.

$A' = k' \otimes_k A$ , still local, max ideal

regular local as $A$ geom reg $\quad \underline{m}' = \{ a' \in A' \mid (a')^p \in m \}$

$k' = A'/m'$ , Since $K'/\mathbb{F}$ is sep we have

$$(*) \quad \begin{tikzcd} 0 \arrow[r] & \underline{m}'/\underline{m}'^2 \arrow[r, "d"] & K' \otimes_{A'} \Omega_{A'/\mathbb{F}} \arrow[r] & \Omega_{K'/\mathbb{F}} \arrow[r] & 0 \\ 0 \arrow[r] & K' \otimes_K (\underline{m}/\underline{m}^2) \arrow[r] \arrow[u] & K' \otimes_A \Omega_{A/\mathbb{F}} \arrow[r] \arrow[u] & K' \otimes_K \Omega_{K/\mathbb{F}} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

$$\dim_{K'} \underline{m}'/\underline{m}'^2 = \dim_{K'} (K' \otimes_K (\underline{m}/\underline{m}^2))$$

JZ of $\mathbb{F} \to K \to K' \Rightarrow$

$$0 \to \Gamma_{K'/K} \to K' \otimes_K \Omega_{K/\mathbb{F}} \to \Omega_{K'/\mathbb{F}} \to \Omega_{K'/K} \to 0$$

By $5.1 \Rightarrow \dim \Gamma_{K'/K} = \dim \Omega_{K'/K}$

Snake of $(*) \Rightarrow \dim(\ker(K' \otimes_A \Omega_{A/\mathbb{F}})$

$= \dim(\operatorname{coker}(-\text{same thing}- \to K' \otimes_{A'} \Omega_{A'/\mathbb{F}}))$

(6)

proof follows after we show

$$\dim\left(\operatorname{coker}\left(K' \otimes_A \Omega_{A/\mathbb{F}} \to K' \otimes_{A'} \Omega_{A'/\mathbb{F}}\right)\right) = n$$

$$A' = A[T_1, \ldots, T_n] \Big/ \underbrace{(T_1^p - x_1, \ldots, T_n^p - x_n)}_{= I}$$

$\Rightarrow$ $d = 0$ in

$$0 \to \Gamma_{A'/A} \to I/I^2 \xrightarrow{d} \bigoplus A' dT_i \to \Omega_{A'/A} \to 0$$

$\Rightarrow$ $\Omega_{A'/A}$ is free on $n$ generators

$\Rightarrow$ $\dim(\operatorname{coker}(-)) = n$.

$\Rightarrow$ $\dim(\ker(-)) = n$

But $dx_1, \ldots, dx_n$ generate this kernel, so they must be lin indep.