[redacted] Zariski's Main Theorem

Thm (ZMT): $A \subset B$ local morphism of local rings essentially finite-type. $B/m_A B$ finite over A/m_A. Then $A \to B$ is essentially finite.

~~We reduce to~~

Lemma (9.2): $A \subset B$ inclusion of quasilocal rings w/ A integrally closed in B. Suppose $A \subset A[t] \subset B$ w/ B essentially finite $/A[t]$. Then if $B/m_A B$ is finite $/ A/m_A$ then $A = B$.

(Here $A[t]$ is not necessarily a polynomial ring i.e. $A[t] \cong A[T]/I$, $T \mapsto t$.) } GLOBAL ASSUMPTION.

Pf (That (9.2) $\Rightarrow$ ZMT). Let $A \subset A[t_1 \cdots t_n] \subset B$ t_i chosen so that B is essentially integral over $A[t_1 \cdots t_n]$.

$n = 0 \Rightarrow$ DONE (ess. finite = ess ft + ess integral).

if $n > 0$, take A' integral closure of $A[t_1, \cdots, t_{n-1}]$ in B

$p = m_B \cap A'$ $A'' = A'_p$.

$A'' \subset A'[t_n] \subset B$ implies by 9.2 that $B = A''$. □

Proceed by induction.

Lemma (9.3): (R, m), ~~integrally closed~~ Quasi-local, m maximal $k = R/m$. $R \subset R[t]$. Then either $R[t] = R_f$ or $R[t] \otimes_R k$ transcendental $/k$.
R int closed in $R[t]$.

Pf: Assume t (in $R[t] \otimes_R k$) not transcendental over k. We have $g \in k[x]$ a monic polynomial for t. Lift g to $R[x]$ and get $g(t) \in m \cdot R[t]$
WLOG, $g(t) = 0$ (by shifting).

deg g cannot be 0.

If $\deg g = 1$, write $g(x) = r + sx$ (both r, s cannot lie in m).

If $s \notin m \Rightarrow t = \frac{r}{s} \Rightarrow R[t] = R$.

If $s \in m$, $r \notin m \Rightarrow$ ~~R[x]/~~ $R[X]/(r+sX) \overset{\varphi}{\twoheadrightarrow} R[t]$

But since $r \in R^\times$, the LHS is just $R_{sr^{-1}}$. $\ker \varphi = 0$ b/c $R \subset R[t]$.

$\deg g = n > 1$. Say $g(x) = ax^n + bx^{n-1} + \cdots$

If $a \notin m$, t is integral $/R$ and $R = R[t]$.

If $a \in m$, at is integral $/R$ and so $at \in R$

If $b \notin m$ and $c = at + b \in m$, we reduce to deg 1 case.

Otherwise, have $(at+b)t^{n-1}$ + lower order $= 0$

proceed by induction.

Def: $A \subset B$ Conductor $\underline{f} = \{a \in A : aB \subset A\}$.

Lem 9.4: $R \subset R[t] \subset D$ R int. closed in D. $D/R[t]$ finite. and $\underline{f}$ conductor of $R[t] \subset D$ $f(x) = ax^n$ + lower terms $\in R[X]$ st $f(t) \in \underline{f}$ Then $a^n \in \underline{f}$

Pf: $a^N \in \underline{f} \Leftrightarrow R[t]_a = D_a$ (conducting commutes w/ localization?)

Thus after localizing at a, WMA f is monic WTS: $R[t] = D$

~~$x \in D$, $g f(t) \in R[t]$~~

let $d \in D$. Then $d f(t) \in R[t]$ b/c $f(t) \in \underline{f}$

Then there is a $g \in R[X]$. w/ $g(t) = d f(t)$.

Divide $g(x) \in R[X]$ by the monic $f \in R[X]$.

$g(X) = f(X) q(X) + r(X)$ $\deg r < \deg f$.

$y = d - q(t)$

$f(t) y = f(t)(d - q(t)) = d f(t) - f(t) q(t)$

$= d f(t) - [g(t) - r(t)]$.

$= r(t)$.

$\bar{D} = D_y$, $\bar{R}, \bar{t}, \bar{y}$ images in $\bar{D}$.

$\bar{f}(\bar{t}) = \bar{y}^{-1} r(\bar{t})$, $\bar{t}$ integral over $\bar{R}[\bar{y}^{-1}]$.

b/c $\deg r < \deg f$.

$\bar{y}$ integral over $\bar{R}[\bar{t}]$ and so over $\bar{R}[\bar{y}^{-1}]$

$\bar{y}^n + a_1 \bar{y}^{n-1} + \text{lower} = 0$.

$y^m (y^n + a_1 y^{n-1} + \text{lower}) = 0$ y integral $/R$.

$d = y + q(t) \in R[t]$.

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Pf of 9.2: $A \subset A[t] \subset C \subset B$. Here,

A int closed in B. $B = C_p$, C finite $/A$.

$\underline{f}$ conductor of $A[t] \subset C$.

Here we are using (9.3)

- $\underline{f} \not\subset P$, take $r \in \underline{f} \setminus p$. $C_r = A[t]_r$, so B is also a localization of $A[t]$. Now, if $A[t]$ is a localization of A, so too is B. But $A \subset B$ is local morphism $\Rightarrow A = B$.
- Otherwise, $B/m_A B$ is a localization of $k[T]$. But B/m_A was finite over $A/m_A = k$. $\rightarrow\leftarrow$.

It therefore suffices to prove that $\underline{f} \not\subset P$.

So assume $\underline{f} \subset P$. Let $Q \subset P^*$ be a minimal prime over $\underline{f}$. Take $q = A \cap Q$, $\bar{A} = A/q$. $\bar{C} = C/Q$.

Have $\bar{A} \subset \bar{A}[\bar{t}] \subset \bar{C}$.

Claim: $\bar{t}$ is transcendental over $\bar{A}$.

If not, $\exists f(x) \in \bar{A}[X]$ w/ $f(X) = aX^n + \dots$ w/ $a \in \bar{A} \setminus q$. $f(t) \in Q$.

As Q minimal over $\underline{f}$ $\exists s \in C - Q$ st $s f(t)^N \in \underline{f}$

So now $f(t)^N$ is in the conductor of $A[t] \subset D = A[t] + sC$

which is finite $/ A[t]$.

~~By (9.4) same f $f \in Q$, $a \notin Q$, $\bar{a} \in \bar{q}$.~~

$(9.4) \Rightarrow a^N \in \underline{f}$ for some N. $\Rightarrow s a^N \in \underline{f}$.

But $\underline{f} \subset Q$ so $a^N \in Q$ (as $s \notin C - Q$).

$\Rightarrow a \in Q$ :: But $a \in A$, so $a \in A \cap Q = \underline{q}$ $\rightarrow\leftarrow$.

(in the noetherian case).

<u>RMK</u>: Zariski's Main Theorem, as stated in Swan's paper follows from the following strong form given in EGA III (Thm 4.4.3).

Let $f: X \to Y$ be quasi-projective w/ X, Y Noetherian. Then U, the set of points $x \in X$ isolated in their fibres is <u>open</u> in X and $f|_U : U \to Y$ factors as an open immersion followed by a finite morphism.

<u>CLAIM</u>: Let $A \to B$ be an essentially finite local morphism of noetherian local rings w/ $B/m_A B$ finite over $k = A/m_A$. Then $A \to B$ is essentially finite.

<u>Pf</u>: Let C be a finite-type A-algebra such that $C_p = B$.

p is simultaneously a minimal and maximal ideal in the fibre $C \otimes_A k$.

$(C \otimes_A k)_p \simeq B/m_A B$, which is artinian

$\Longrightarrow$ minimality of p.

The residue field $k(P)$ of P in C is a quotient of $B/m_A B$ and has transcendence degree $0 \Rightarrow P$ is maximal.

Thus, P is isolated in its fibre. By Grothendieck's thm:

$$\begin{array}{ccc} \operatorname{Spec} C & \xrightarrow{\ f\ } & \operatorname{Spec} A \\ \uparrow {\scriptstyle \text{open immersion}} & & \nearrow {\scriptstyle f' \text{ finite}} \\ p \in U & \xrightarrow[\text{open immersion}]{} & Z \end{array}$$

f finite type + affine $\Rightarrow$ quasi-projective

Z is finite / $\operatorname{Spec} A$ and hence affine $Z = \operatorname{Spec} D$

D/A finite $B = C_p$ is just a localization of D.

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