Usual statement: (R,m) local (S,n) local Noeth. R-alg s.t. $mS \subseteq n$.
M f.g. S mod $\Rightarrow$ M flat as R-mod $\Leftrightarrow$ $\mathrm{Tor}_1^R(R/m, M) = 0$.

Page 1

state & prove

Recall: can check flatness by checking $I \otimes M \hookrightarrow M$ injections $\forall$ ideals $I \subseteq R$. this stated for max'l gives only have to check this on max'l ideal jacobson rad. of A.

★ Thm 7.1: (Local criterion of flatness)
Let $R \to A$ a map b/w Noeth. rings, w/ $IA \subseteq J(A)$
Let M f.g. A mod. Then M flat $/R$ $\Leftrightarrow$ M/I^nM flat over R/I^n $\forall n$.

$(\longrightarrow)$ Clear (will show $\forall$ s.es. $\otimes_R M$ is exact)
$(\longleftarrow)$ Consider a s.e.s
$$0 \to N' \to N \to N'' \to 0$$

Note: Artin-Rees gives us: $N' \cap I^n N = I^{n-r}\underbrace{(I^r N \cap N')}_{\text{call } N_0'}$ some r, $n \geq r$

by $\otimes R/I^n$ we get a seq:
$$0 \to N'/I^{n-r}N_0' \to N/I^nN \to N''/I^nN'' \to 0$$

$\otimes_{R/I^n} M/I^nM$:
$$0 \to M \otimes_R (N'/I^{n-r}N_0') \to M \otimes_R (N/I^nN)$$
(Note $M \otimes_R (N'/I^{n-r}N_0') \cong M/I^nM \otimes_{R/I^n} (N'/I^{n-r}N_0')$)
Let $L = \mathrm{im}(M \otimes_R N_0' \to M \otimes N')$
then $I^{n-r}L = \mathrm{im}(M \otimes_R I^{n-r}N_0' \to M \otimes_R N')$ (clear think about it)
$\Rightarrow (M \otimes_R N')/I^{n-r}L = M \otimes_R (N'/I^{n-r}N_0')$
$\Rightarrow 0 \to (M \otimes N')/I^{n-r}L \to (M \otimes_R N)/I^n(M \otimes_R N)$ is exact.
$\Rightarrow \ker(M \otimes_R N' \to M \otimes_R N) \subseteq \bigcap_{n>r} I^{n-r}L = 0$
L f.g A mod & $IA \subseteq J(A)$

state?

Lemma 7.2 R, A, M, I as above. $\mathrm{Tor}_1^R(M,N) = 0$ $\forall$ Amod N
$\Leftrightarrow$ $M \otimes_R A$ flat & $\mathrm{Tor}_1^R(M,A) = 0$.

Don't prove: $(\Leftarrow)$ If $\mathrm{Tor}_1^R(M,A) = 0$ $\forall$ A-mod N, then $M \otimes_R N = M \otimes_R A \otimes_A N$ is exact in $N \Rightarrow M \otimes_R A$ flat over A.
$(\Leftarrow)$ Let N be any A-mod, $0 \to L \to F \to N \to 0$ w/ F free over A. Then, since $M \otimes_R N = M \otimes_R A \otimes_A N$ is exact in N, $0 = \mathrm{Tor}_1^R(M,F) \to \mathrm{Tor}_1^R(M,N) \to M \otimes_R L \to M \otimes_R F$
so $\mathrm{Tor}_1^R(M,N) = 0$

$\boxed{\text{sketch: use fact } M \otimes_R N = M \otimes_R A \otimes_A N}$

state ~~(& prove?)~~

Corollary 7.3 R, A, M, I as above. M is flat over $R \Leftrightarrow$ M/IM flat over R/I & $\mathrm{Tor}_1^R(R/I, M) = 0$.

($\rightarrow$) ✓

($\leftarrow$) 7.2 w/ $A = R/I$ gives $\mathrm{Tor}_1^R(M,N) = 0$ for R/I mods N (since $M/IM = M \otimes_R R/I$ flat over R/I & $\mathrm{Tor}_1^R(M, R/I) = 0$ by assumption)

So $\mathrm{Tor}_1^R(M,N)=0$ if $IN = 0 \Rightarrow$ also if $I^n N = 0$ by l.e.s in Tor for some n. Then $\mathrm{Tor}_1^R(M,N) = 0$ for R/I^n-mods $\Rightarrow$ $M \otimes R/I^n = M/I^nM$ is flat over R/I^n so by Thm 7.1 M is flat over R

$\boxed{\text{Sketch: use 7.2 + 7.1}}$

state/sketch

Lem 7.4 $R \to A \to B$ A, B Noetherian. $I \subseteq R$ s.t. $IB \subseteq J(B)$ B flat / R & B/IB flat / A/IA $\Rightarrow$ B flat / A.

Pf: Apply 7.3 to $A \to B$ & IA. B is flat over A iff B/IB is flat over A/IA & $\mathrm{Tor}_1^A(A/IA, B) = 0$.

use

$$0 \to IA \to A \to A/IA \to 0 \qquad \otimes B$$

wts $\mathrm{Tor}_1^A(A/IA, B) = 0$ but this happens if

$$\cancel{\mathrm{Tor}_1^A(A,B)} \to \underset{0}{\overset{}{\mathrm{Tor}_1^A(A/IA, B)}} \to IA \otimes B \to B \to A/_{IA} \otimes B \to 0$$

if $IA \otimes B \hookrightarrow A \otimes B \longrightarrow A/IA \otimes B$

$$\qquad\qquad\qquad\qquad B \qquad\qquad\quad B/IB$$

don't prove

$\Rightarrow$ if $IA \otimes B \simeq IB$.

B flat over $R \Rightarrow \mathrm{Tor}_1^R(R/I, B) = 0$ so $I \otimes_R B \to IB$

but then have $\underbrace{I \otimes_R B = I \otimes_R A \otimes_A B \twoheadrightarrow IA \otimes_A B \twoheadrightarrow IB}_{\sim}$

so all must be iso.

$\boxed{\text{Sketch: use 7.3 w/ } A \to B \text{ \& } IA \text{ \& fact that } B \text{ flat}/R}$

don't state? or prove

Lem 7.5 Let $A \to B$ & let $x_1, \ldots, x_n \in A$ be a reg seq on A & on B. Then $Tor_1^A(A/(x_1,\ldots,x_n)A, B) = 0$

Pf clear for n=1! Let $I = (x_1, \ldots, x_{n-1})A$ & $x = x_n$. Assume works $< n$. By induction on n, $Tor_1^A(A/I, B) = 0$. Then $0 \to A/I \xrightarrow{x} A/I \to A/(I,x)$ gives $0 \to Tor_1^A(A/(I,x), B) \to B/IB \xrightarrow{\cdot x} B/IB$ but last map inj.

$\boxed{\text{sketch: induct on } n.}$

Lemma 7.6 Let $A \to B$ be a local morph of reg rings. Suppose $B \otimes_A m_A/m_A^2 \to m_B/m_B^2$ is inj. (so that a reg sys. of parameters of A maps to part of a reg. sys. of param of B). Then B flat / A & $B/m_A B$ is reg of dim dim B − dim$A$.

State

Pf: Apply Cor 7.3 to $A \to B$ w/ $I = m_A$.
Have: $B/m_A B$ flat over A/m_A & $Tor_1^A(A/m_A, B) = 0$
$\Rightarrow$ B flat over A.
Just check $Tor_1^A(A/m_A, B) = 0$, but this is 7.5

$\boxed{\text{Sketch: use 7.3 \& 7.5}}$

skip?

Lem. 7.7 Let $A \to B$ flat local morphism of local rings. Say $B/m_A B$ is reg of dim d. Let $\varphi: A[x_1, \ldots x_d] \to B$ w/ x_i mapping to part of a reg sys. of param of $B/m_A B$. Let $P = \varphi^{-1}(m_B)$. Then $A[x_1 \ldots x_n] \to B$ is flat so φ is flat & B/PB is reg of dim $d-n$.

Pf Apply 7.4 to $A \to A[x]_P \to B$: suffices to check $B/m_A B$ is flat over $A[x]_P/m_A$. ~~AST~~ Work mod m_A $\Rightarrow$ assume $A = k$. a field. $\varphi: k[x_1, \ldots x_n] \to B$ so $\varphi^{-1}(m_B) = (x_1, \ldots x_n)$ so $k[x_1 \ldots x_n] \to B/m_A B$ sat. 7.6 & so B/m_A flat / ~~[illegible]~~ $k[x_1 \ldots x_n]$ as desired. ✓

$\boxed{\text{Sketch: use 7.4 on } A \to A[x_1 \ldots x_n]_P \to B \text{ to work mod } m_A \text{. apply 7.6}}$

skip?

Lem 7.8 $A \to B$ flat local morphism of local rings w/ residue class fields $F \subseteq K$. Let ~~[illegible]~~ $\varphi: A[x_1, \ldots, x_n] \to B$ send x_i to elts whose images in K are alg. indep over F. Then φ is flat.

PF: Let $P = \varphi^{-1}(m_B)$. Lemma 7.4 on $A \to A[x_1, \ldots x_n]_P \to B$ $\Rightarrow$ can work mod m_A. $\Rightarrow$ can assume $A = F$ so that $P = 0 \Rightarrow A[x_1 \ldots x_n]_P$ is a field $\to B$ flat over it.

proof

Lemma 7.9 Let $A \to B$ flat local morph. of local rings. Suppose $B/m_A B$ is geom. reg of dim d over $k = A/m_A$.
Let $K = B/m_B$ & let $k \subseteq E \subseteq K$ w/ E field of fin type$/k$.
Let $y_1 \ldots y_s \in B$ map to a p-base $\bar{y}_1, \ldots \bar{y}_s$ for E over k.
Let $\varphi : A[Y_1 \ldots Y_s] \to B$ by sending $Y_i \to y_i$. Let $P = \varphi^{-1}(m_B)$. Then $A[Y_1, \ldots Y_n]_P \to B$ flat. In part.
φ flat & B/PB is reg of dim $= \dim B - \dim_E \Gamma_{E/k} = \dim B - \dim_L \Gamma_{L/k}$

Pf Apply lemma 7.4 to $A \to A[Y_1 \ldots Y_n]_P \to B$
$\Rightarrow$ proving last map flat equiv. to proving it flat mod m_A $\Rightarrow$ work mod m_A w/ $L = k(\bar{y}_1, \cdots \bar{y}_n) \subseteq E$

Let $C = k[Y]_P$ so $C/m_C = L = k(\bar{y}_1, \ldots \bar{y}_n)$
C is essentially smooth over k $\Rightarrow$ $\Gamma_{C/k} = 0$ $\Omega_{C/k}$ projective
JZ seq of: $k \to C \to L$
is:
$$0 \to \Gamma_{L/k} \to \Gamma_{L/C} \to L \otimes_C \Omega_{C/k} \to \Omega_{L/k} \to 0$$
(The dy_i generate $\Omega_{C/k}$ & we've ensure their images will be linearly indep in $\Omega_{E/k}$ $\Rightarrow$ in $\Omega_{L/k}$
$\Rightarrow$ $L \otimes \Omega_{C/k} \xrightarrow{\sim} \Omega_{L/k}$
$\Rightarrow$ $\Gamma_{L/k} \xrightarrow{\sim} \Gamma_{L/C}$ but $\Gamma_{L/C} = m_C/m_C^2$ & Krull dim $C = \dim_k m_C/m_C^2 = \dim_k \Gamma_{L/k}$.

Consider:
$$\begin{array}{ccccccc} & K \otimes_L \Gamma_{L/k} & \xrightarrow{\sim} & K \otimes_L \Gamma_{L/C} & = & K \otimes_L (m_C/m_C^2) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \Gamma_{K/k} & \to & \Gamma_{K/B} & = & m_B/m_B^2 \end{array}$$

Wenliang: Thm 6.1 $\Rightarrow$ $\Gamma_{K/k} \hookrightarrow \Gamma_{K/B}$
(Thm 6.1: k field, A geo. reg local k alg $K = A/m_A$
$0 \to \Gamma_{K/k} \to m_A/m_A^2 \xrightarrow{d} K \otimes \Omega_{A/k} \to \Omega_{K/k} \to 0$ exact)

Wenliang: Cor 5.2 $\Rightarrow$ $K \otimes_L \Gamma_{L/k} \hookrightarrow \Gamma_{K/k}$
(Cor 5.2: $F \subseteq E \subseteq K$ ext. of fields $\Rightarrow$ JZ seq is $0 \to K \otimes_E \Gamma_{E/F} \hookrightarrow \Gamma_{K/F} \to \Gamma_{K/E} \to K \otimes_E \Omega_{E/F} \to \Omega_{K/F} \to \Omega_{K/F} \to 0$)

$\Rightarrow$ $K \otimes_L \Gamma_{L/C} \hookrightarrow \Gamma_{K/B}$ & $K \otimes_L (m_C/m_C^2) \hookrightarrow m_B/m_B^2$

$\Rightarrow$ B flat over C by 7.6.
& also 7.6 gives: $\dim B/pB = \dim B - \dim C = \dim B - \dim_L \Gamma_{L/k}$

JZ seq of $k \subseteq L \subseteq E$:

$$E \otimes_L \Gamma_{L/k} \to \Gamma_{E/k} \to \Gamma_{E/L} \to E \otimes_L \Omega_{L/k} \to \Omega_{E/k} \to \Omega_{E/L} \to 0$$

but $E \otimes_L \Omega_{L/k} \twoheadrightarrow \Omega_{E/k}$
$\Rightarrow \Omega_{E/L} = 0$.

Wenliang

Cor 5.3 $\Rightarrow$ E/L finite, separable
(Cor 5.3: If E/F field ext of fin. type, then $\Omega_{E/F} = 0$ iff E/F fin. sep.)
$\Rightarrow \Gamma_{E/L} = 0$ so

$$E \otimes_L \Gamma_{L/k} \xrightarrow{\sim} \Gamma_{E/k}$$
$$\Rightarrow \dim_L \Gamma_{L/k} = \dim_E \Gamma_{E/k}$$

__CM rings__

Recall: $\operatorname{depth}_I M = \min\{n \mid \operatorname{Ext}_R^n(R/I, M) \neq 0\}$
$=$ max'l length of a reg seq on M in I

Equiv b/c
- $\operatorname{depth}_I M = 0 \Leftrightarrow \operatorname{Hom}_R(R/I, M) \neq 0$
  $\Leftrightarrow$ I consists of zero divisors on M
- l.e.s in Ext associated to
  $0 \to M \xrightarrow{x} M \to M/xM \to 0$
  gives $\operatorname{depth}_I M/xM = \operatorname{depth}_I M - 1$

Note: immediately get $I \subseteq J \Rightarrow \operatorname{depth}_I M \leq \operatorname{depth}_J M$.

state:
__Lemma 8.1__ $\operatorname{depth}_I M = \min_{P \supseteq I} \operatorname{depth}_P M$ over prime ideals $P \supseteq I$

sketch (R/I has a finite filtration w/ quotients of the form R/P so look at l.e.s. in Ext)

__Def__ The __grade__ of I is $\operatorname{gr} I = \operatorname{depth}_I R$

Lemma 8.2 $\operatorname{gr} I \leq \operatorname{ht} I$

$\operatorname{ht} I = \min_{P \supseteq I} \operatorname{ht} P$ so w/ 8.1 can assume $I = P$.

localizing preserves reg seq $\Rightarrow$ $\operatorname{gr} P \leq \operatorname{gr} P_P$ $\Rightarrow$ assume P max'l
If $n = \operatorname{gr} P$ w/ $x_1, \dots x_n$ reg seq then $\dim R/(x_1 \dots x_n) = \dim R - n \geq 0$
$\Rightarrow$ $n \leq \dim P = \operatorname{ht} P$.

Def R CM if $\operatorname{gr} I = \operatorname{ht} I \;\; \forall I$

Note: 1) Can check this on primes
2) $\operatorname{gr} P_S \geq \operatorname{gr} P$ but $\operatorname{ht} P_S = \operatorname{ht} P$ $\Rightarrow$
R CM $\Rightarrow$ R_S is CM.

Prove if time?

Lemma 8.3 If P is prime but not max'l $\operatorname{depth}_P M \geq \min_{Q \supsetneq P} \operatorname{depth}_Q M - 1$ over primes $Q \supsetneq P$.
$\Rightarrow$ $\operatorname{gr} P \geq \min_{Q \supsetneq P} \operatorname{gr} Q - 1$

Pf: Pick $a \in P$. ~~Ext~~ $0 \to R/p \xrightarrow{\cdot a} R/p \to R/(P,a) \to 0$
gives l.e.s. $\operatorname{Ext}^n_R(R/p, M) \xrightarrow{\cdot a} \operatorname{Ext}^n_R(R/p, M) \to$
$\operatorname{Ext}^{n+1}(R/(P,a), M) \to \dots$

Last term $= 0$ if $n+1 < \min_{Q \supsetneq P} \operatorname{depth}_Q M$

(Since $\operatorname{depth}_{(P,a)} M = \min_{Q \supseteq (P,a)} \operatorname{depth}_Q M \geq$ ~~depth~~ $\min_{Q \supsetneq P} \operatorname{depth}_Q M$)

Let $I = \operatorname{Ann} \operatorname{Ext}^n_R(R/p, M)$.
$\Rightarrow$ $I + aR = R$. If $I \supsetneq P$ let $a \in I - P$
$\Rightarrow$ $a \in I$ so $I = R$. If $I = P$ then
$P + aR = R \;\; \forall a \notin P$ $\Rightarrow$ P max'l.

~~Lemma~~ Cor 8.4: TFAE: 1) R is CM
2) $\operatorname{gr} m = \operatorname{ht} m \;\; \forall$ max'l ideals m of R
3) R_m is CM $\forall$ max'l ideals of R

Pf 3) $\Rightarrow$ 2) $\Rightarrow$ 1) $\Rightarrow$ 3).
3) $\Rightarrow$ 2) $\operatorname{Ext}^n_R(R/m, R)$ is ann. by m $\Rightarrow$ localizing at m doesn't change it's vanishing
2) $\Rightarrow$ 1) use prev. lemma
1) $\Rightarrow$ 3) clear.

state & prove

Cor 8.5 If R CM & P prime then all max'l chains of primes $P = P_0 \supseteq \cdots \supseteq P_n$ have same length.
Pf: Show if $P \supseteq Q$ w/ no prime in between $\Rightarrow ht\,P = ht\,Q + 1$
Localize @ P. 8.3 $\Rightarrow$ $gr\,Q \geq gr\,P - 1 \Rightarrow ht\,Q \geq ht\,P - 1$

skip

Lemma 8.6: R local, CM x nonunit $\rightarrow$ $\dim R/(x) = \dim R - 1 \Leftrightarrow$ x reg.
Pf Clear if x reg. If x not reg let P be minimal over (x). Then $gr\,P = 0 \Rightarrow ht\,P = 0$. Cor 8.5 $\Rightarrow \dim R/P = \dim R$
$\Rightarrow \dim R/(x) = \dim R$

skip

Cor 8.7 R local CM & $x_1, \ldots x_n$ sys of param $\Rightarrow$ $x_1 \ldots x_n$ is a reg sequence
Pf: $\boxed{\text{Sketch: induct \& use 8.6}}$
($\dim R/(x_1, \ldots x_i) = \dim R - i \Rightarrow x_{i+1}$ reg on $R/(x_1, \ldots x_i)$ by 8.6)

skip

Lemma 8.8 If R is CM then so is R[x]
Pf: ~~Assume R local w/ max'l ideal P. k = R/P.~~
Let m be a max'l ideal of R[x]. $P = m \cap R$.
$R[x]_m$ is a localization of $R_P[x]$ $\Rightarrow$ replace R by R_P & say R local w/ max'l ideal P. Let $k = R/P$.
then $m/P[x]$ is max'l in $k[x]$ $\Rightarrow$ gen. by monic $\bar{f}$.
Lift to f over R. Then $m = (P, f)$. Let $x_1 \ldots x_n$ be reg seq in P on R. Then $f, x_1 \ldots x_n$ reg on R[x] b/c $R[x]/(f)$ free R-mod. $\Rightarrow gr\,m \geq 1 + gr\,P = 1 + ht\,P = ht\,m$

state & prove

Thm 8.9 $A \subseteq B$ int ext. then P lies over p iff P max'l wrt property $P \cap A \subseteq p$.

$$\begin{array}{c} P \subseteq B \\ | \quad\ \cup\!\!\!| \\ p \subseteq A \end{array}$$

Pf ($\longrightarrow$) Suppose P prime & lies over p $\Rightarrow$ $P \cap A = p$.
~~Work mod P so assume P = p = 0.~~ wts $\nexists$ I s.t. $I \supseteq P$ & $I \cap A = p$. Work mod P so assume $P = p = 0$.
let $x \in I$ $x \neq 0$. $x \in B$ so x sat.
$x^n + a_1 x^{n-1} + \cdots + a_n = 0$ $a_i \in A$ & choose n minimal.
$\Rightarrow a_n \neq 0$ but $a_n \in I \cap A \Rightarrow I \cap A \neq 0 = p$. $\times$.

($\Leftarrow$) Say P max'l s.t. $P \cap A \subseteq p$. Then P prime b/c P max'l wrt $P \cap S = \emptyset$ w/ $S = A - p$ mult. set.

Work mod P. $\Rightarrow$ $P = 0$.

We've assumed $I \cap A \subseteq p \Rightarrow I \subseteq P$. Let $x \in p$.

Let $b \in B$ sat. $b^n + a_1 b^{n-1} + \cdots + a_n = 0$ $a_i \in A$.

Assume $bx \in A \Rightarrow (bx)^n + ax(bx)^{n-1} + \cdots + a_n x^n = 0$

$\Rightarrow (bx)^n \in Ax \Rightarrow \in p \Rightarrow bx \in p$

$\Rightarrow Bx \cap A \subseteq p \Rightarrow x = 0 \Rightarrow P \cap A = 0 = p$.

write

Cor 8.10 (Lying over) $A \subseteq B$ integral $\Rightarrow$ $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$
(Can always find $P \subseteq B$ lying over $p \subseteq A$.

Cor 8.11 (Going up) $A \subseteq B$ integral. P prime ideal of B. $p = P \cap A$ & ~~$p \subseteq q$~~ $p \subseteq q$ some $q \subseteq A$. Then $\exists\, Q \subseteq B$ w/ $q = Q \cap A$ & $P \subseteq Q$

$$\begin{array}{ccccc} P & \subseteq & Q & \subseteq & B \\ | & & \exists \vdots & & \cup \\ p & \subseteq & q & \subseteq & A \end{array}$$

Pf. Given 8.9, can pick Q max'l s.t. $Q \cap A \subseteq q$ & will clearly contain P (since $P \cap A \subseteq p \subseteq q$) $\Rightarrow$ Q will lie over q.

~~Thm 8.12 – see next page?~~

Thm 8.12 (Going down thm) ~~A⊆B~~ $A \subseteq B$ integral. A normal ~~A~~ domain, B torsion free as A-mod. Let $P \subseteq B$, $p = P \cap A$. ~~...~~ $p \supseteq q$

$$\begin{array}{ccccc} Q & \subseteq & P & & B \\ | & & | & & \cup| \\ q & \subseteq & p & \subseteq & A \end{array}$$

Pf (Cohen-Seidenberg)

Lemma 8.13 A normal domain q. field K. B A-alg torsion free as A-mod. $b \in B$ integral over A. Let $f(x)$ minimal monic eq. of $b \in KB$ over K. Then $\{g(x) \in A[x] \mid g(b) = 0\} = A[x] \cdot f(x)$. $\Rightarrow f \in A[x]$

Pf $f(b) = 0$. If $g(b) = 0$ then $g = fh$ in $K[x]$. Say $g \in A[x]$ monic. (Possible since b integral over A) $\Rightarrow$ roots of g are integral over A $\Rightarrow$ same is true of f & h. $\Rightarrow$ coeff of ~~B~~ f & h are int. over A $\Rightarrow \in A$ which is normal. If g not monic apply this argument to $g + f^N$ $N \gg 0$.

Lemma 8.14 $A \subseteq B$ integral ext. & $I \subseteq A$ ideal. Then $\sqrt{IB}$ is $\{x \in B$ sat. some $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ w/ all $a_i \in I\}$.

Pf $\supseteq$ clear

$x \in B$, $a_i \in I$

$\subseteq$ Let $y = x^N \in IB$. $\exists$ a subring $C = \sum A w_i$ of B, fin over A w/ $y \in IC$. Let $y w_i = \sum q_{ij} w_j$ w/ $q_{ij} \in I$. Then $|yI - (q_{ij})| w_i = 0$. so $|yI - (q_{ij})| = 0$ w/c gives eq. for y & $x^N = y$ gives eq. for x.

Pf (8.12) Let $S = (B-P)(A-q)$ wts $S \cap Bq = \emptyset$ & take $Q \supseteq Bq$ max'l s.t. $Q \cap S = \emptyset$. (& then will have Q max'l w/ $Q \subseteq P$ $Q \cap A \subseteq q$?) Suppose for cont. $Q \cap S \neq \emptyset$. Take $ab \in Bq$ w/ $a \in A - q$ & $b \in B - P$. Let $f(b) = 0$, $f = x^n + c_1 x^{n-1} + \ldots c_n$ w/ $c_i \in A$ is minimal eq of b over A. Then $h(ab) = 0$ where $h(x) = x^n + ac_1 x^{n-1} + \ldots + a^n c_n$ is minimal eq. for ab over A.

Since $ab \in Bq$ 8.14 $\Rightarrow$ ab sat some monic g. $g(ab) = 0$ w/ $\bar{g}(x) = x^n$ (mod q). By 8.13 $h \mid g$ $\Rightarrow$ ~~...~~ $\bar{h} \mid \bar{g}$ $\Rightarrow \bar{h} = x^n$.

$\Rightarrow a^i c_i \in q$ $\forall i$. Since $a \in A - q$ all $c_i \in q$ $\Rightarrow \bar{f} = x^n$.

8.14 $\Rightarrow$ $b \in \sqrt{qB} \subseteq P$ $\times$.

Def $q \subseteq p$ in A sat. going down wrt B if $P \subseteq B$ over p contains a Q over q

skip

Cor 8.15 A domain, A' int. closure of A in quot. field. If $q \subseteq p$ in A sat going down wrt A', then it sat. going down wrt any $B \supseteq A$ w/ B int. over A & torsion free as an A mod.

Pf Let $P \cap B = p$. Let K be quot. field of A. Since B is torsion free over A, B is a subring of KB & we can construct $A'B$ inside of KB. Since $A'B$ int. over B $\exists$ prime P' of $A'B$ over P. Let $p' = P' \cap A'$. This lies over p so by hyp. $\exists$ prime $q' \subseteq p'$ lying over q. Now $A' \subseteq A'B$ sat. hyp. of 8.12 $\Rightarrow$ $\exists$ Q' of $A'B$ over q' w/ $Q' \subseteq P'$. Let $Q = B \cap Q'$.

say write only

Cor 8.16 R domain & $A = R[X]$ poly ring over R. Let p be prime ideal of A, set $p_0 = R \cap p$ & $q = p_0 A$. Then $q \subseteq p$ sat. going down wrt any $B \supseteq A$ w/ B int over A & torsion free as an A mod.

Pf: By Cor 8.15 can consider $B = A' = R'[X]$. Let p' be a prime of A' over p & let $p_0' = R' \cap p'$. Then $q' = p_0' A'$ is the required prime