Recall: $A = R[X_1 \cdots X_n]/I$

- $H_{A/R} = \text{rad}\left(\sum_{(f_1 \cdots f_p) \subset I} \Delta(f_1, \cdots f_r)[(f_1 \cdots f_r) : I] \right) \subset A.$

Complement is the smooth locus of A/R.

- $a \in H_{A/R}$ is standard if $a^N \in \Delta(f_1 \cdots f_r)[(f_1 \cdots f_r):I]$ for some $f_1 \cdots f_r$, $N \in \mathbb{N}$
  —"— strictly standard if $a \in \Delta(f_1, \cdots f_r)[(f_1 \cdots f_r):I]$

Given $R \to A \to \Lambda$ we define $h_A = \text{rad}(H_{A/R}\Lambda)$.

Thm: Suppose $R \to \Lambda$ is geom. reg. map of Noeth. rings. Let A/R f.t w/ $R \to A \to \Lambda$. Then $\exists\ B/R$ smooth st $R \to A \to B \to \Lambda$.

Lem 11.1: If $h_A = \Lambda$, the above theorem is true.

PF: Let $H_{A/R} = (a_1 \cdots a_n)$. Then $\exists \lambda_i \in \Lambda$ st. $\sum a_i \lambda_i = 1$. Put $B = A[X_1 \cdots X_n]/(\sum a_i X_i - 1)$

Inverting a_i allows us to solve $\sum a_i X_i - 1 = 0$ for X_i so $B_{a_i} = A_{a_i}[X_1 \cdots \widehat{X_i} \cdots X_n]$ R-smooth.

$D(a_i)$ covers spec B. as $V(H_{A/R}B) = \emptyset \Rightarrow B/R$ smooth. $\square$

Idea: Given $R \to A \to \Lambda$, we shall construct $R \to A \to B \to \Lambda$ w/ $h_A \subsetneq h_B$. By Noetherian induction, we will eventually get $h_B = (1)$

$\to$

RMK: Let $R \to A \to \Lambda$ be as before, $\overset{\varphi}{\curvearrowright}$ from R to Λ. $I = \varphi^{-1}(h_A)$.

h_A = radical ideal, so $h_A = P_1 \cap \cdots \cap P_K$ P_i minimal over h_A.

$$I = \varphi^{-1}(P_1) \cap \cdots \cap \varphi^{-1}(P_K). \Rightarrow \cancel{V(I) \subset \bigcup V(\varphi^{-1}(P_i))}.$$

Let q be minimal over I. Then $\varphi^{-1}(P_1) \cap \cdots \cap \varphi^{-1}(P_K) \subset q$

$V(q) \subset \bigcup V(\varphi^{-1}(P_i)). \Rightarrow \underset{\text{by irreducibility of } V(q).}{V(q) \subset V(\varphi^{-1}(P_i))}$ some i.

$\varphi^{-1}(P_i) \subset q$ so by minimality they are equal.

Hence, given $R \to A \to \Lambda$, we may assume that we can choose P minimal over h_A which contracts to q which is minimal over $\varphi^{-1}(h_A)$.

It suffices to prove:

<u>Thm</u>$^{11.2}$: Let $R \to A \to \Lambda$ w/ A f.t $/R$. Let P be minimal over h_A and say that $\varphi^{-1}(P) = \mathfrak{q}$ which is minimal over $\varphi^{-1}(h_A)$. Suppose that $k(\mathfrak{q}) \to (k(\mathfrak{q}) \otimes_R \Lambda)_p$ is geometrically reg, ~~Aq~~ $R_\mathfrak{q} \to \Lambda_p$ flat. Then there is B f.t. $/R$ w/ $R \to A \to B \to \Lambda$, $h_A \subset h_B$ $h_B \not\subset P$.

⌈<u>RMK</u>: We have relaxed $R \to \Lambda$ geom reg to $R_\mathfrak{q} \to \Lambda_p$ $\mathfrak{q}R_\mathfrak{q}$-smooth ⌋

<u>Def</u>: Given input data $R \to A \to \Lambda \supset P$ w/ $h_A \subset P$ minimal We shall say that "$R \to A \to \Lambda \supset P$" is <u>Resolvable</u> if we can find a B as in the thm.

<u>Strategy</u>: We shall induct on $\text{ht}\, P$. For a ~~suitably chosen~~ any $J \subset R$. Note that if $R_\mathfrak{q} \to \Lambda_p$ is flat w/ geom reg fibre, then ditto for $R_\mathfrak{q}/JR_\mathfrak{q} \to \dfrac{\Lambda_p}{J\Lambda_p}$. For a well-chosen J, $\text{ht}\, P > \text{ht}(P/J\Lambda)$. $\bar{R} = R/J$ and resolvability of $\bar{R} \to \bar{A} \to \bar{\Lambda} \supset \bar{P}$ $\Rightarrow$ resolvability of $R \to A \to \Lambda \to P$.

<u>Lemma 11.3/11.4</u> (Main technical tool) Let $R \to A \to \Lambda$. $a \in R$. Suppose

1) $\text{Ann}_R(a) = \text{Ann}_R(a^2)$; $\text{Ann}_\Lambda(a) = \text{Ann}_\Lambda(a^2)$.
2) ~~a~~ image of a in A is <u>strictly standard</u>.

Let $c \geq 8$ $\bar{R} = R/a^cR$ $\bar{A} = A/a^cA$ etc. ~~Let~~ Then if $\bar{R} \to \bar{A} \to \Lambda \supset \bar{P}$ is resolvable then so too is $R \to A \to \Lambda \supset P$.

<u>Standardization</u>: How do we arrange for conditions ① and ② to hold?

① is easy $\text{Ann}_R(a) \subset \text{Ann}(a^2) \subset \cdots$ so since all rings are noetherian the chain stabilizes

② Is much more serious.

<u>Lemma</u> (11.6/11.7) Let $R \to A \to \Lambda$ A/R ft. $a \in R$. such that $a \in h_A$. Then there is a ft R-alg C w/ $R \to A \to C \to \Lambda$ such that $a \in C$ is standard. (wrt some presentation) And $H_{A/R} \subset H_{A/R}C$

(so $h \subset h$)

Pf Sketch: Put $H_{A/R} = (b_1, \dots b_n)$.

$\forall\, a \in h_A = \sqrt{H_{A/R}\Lambda} \Rightarrow \exists \lambda_i \in \Lambda$ st $a^N = \sum \lambda_i b_i$ for some N.

$$B = A[x_1 \cdots x_n] \Big/ (\sum b_i x_i - a^N) \longrightarrow \Lambda \quad (x_i \mapsto \lambda_i)$$

Inverting b_i means we can solve $\sum b_i x_i - a^N = 0$ for $x_i \Rightarrow$

$B_{b_i} = A_{b_i}[x_1 \cdot\cdot \hat{x_i} \cdot\cdot x_n] \leftarrow$ Smooth over $A_{b_i} \leftarrow$ smooth over R.

$D(b_i) \subset$ smooth locus of $B = V(H_{B/R})^c$

so

$\Rightarrow V(b_i) \supset V(H_{B/R}) \Rightarrow b_i \in \sqrt{H_{B/R}} = H_{B/R}$. $(H_{A/R} B \subset H_{B/R}$

Thus $a^N \equiv \sum b_i x_i \in H_{B/R}$.

~~Step 2: Write $B = R[\underline{x}]/I$ $C = Sym(I/I^2)$~~

~~Step: $0 \to I_a \to R_a[\underline{x}]/I_a \to B_a \to 0$ B_a/R_a is smooth~~

~~$\Rightarrow$ Conormal sequence split exact.~~

~~$0 \to (I/I^2)_a \to \Omega_{R_a[x]/R_a} \otimes B_a \to \Omega_{B_a/R_a} \to 0$~~

~~C_a is a polynomial over $B_a \Rightarrow$ R-smooth.~~

Step 2: Put $B = R[\underline{X}]/I$. $C = Sym(I/I^2)$.

Let $b \in H_{B/R}$. Conormal sequence

$$0 \to (I/I^2)_b \to \Omega_{R[\underline{x}]/R} \otimes B_b \to \Omega_{B_b/R} \to 0$$

Split exact b/c B_b/R smooth. $\Rightarrow$ ~~C_b polynomial~~ $(I/I^2)_b$ loc free

C_b is R-smooth. $\Rightarrow D(b) \subset$ R-smooth locus of C

$\Rightarrow b \in H_{C/R}$

$\Rightarrow H_{B/R} C \subset H_{C/R}$.

~~[illegible]~~ Hence $a \in H_{C/R}$ and w/ some checking can be shown to be standard [Sw 5.4]

We have

$$\begin{tikzcd} R \arrow[r] & A \arrow[r] & B \arrow[r, "i"'] \arrow[d, "\varphi"'] & C \arrow[l, bend right, "s"'] \arrow[dl, "\varphi s"] \\ & & \Lambda & \end{tikzcd}$$

use section to map C to Λ. □

Reduction to the Artinian Case

<u>Lem</u>: 12.1

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend right] & A \arrow[r] & \Lambda \supset P \end{tikzcd}$$

w/ q minimal over $\varphi^{-1}(h_A)$.

φ It suffices to show resolvability for $\text{ht}\, q = 0$.

<u>Pf</u>

Suppose $\text{ht}\, q > 0$. Choose $a \in I \subset q$ so that a is not in any height 0 prime. Put $\bar{q} = q(R/aR)$. Then $\text{ht}\, q > \text{ht}\, \bar{q}$.

By <u>standardization</u> $\exists\, C/R$ w/ $h_A \subset h_C$ and a standard in C. $[R \to A \to C \to \Lambda$

If $h_C \not\subset P$ then the resolution is complete. Otherwise,

For some N a^N satisfies ① & ② of <u>main</u> Lemma. Take $\bar{R} = R/a^{8N}$ etc.

Then $\bar{R} \to \cancel{\bar{A}} \to \bar{C} \to \bar{\Lambda} \supset \bar{P}$ is resolvable by induction on $\text{ht}\, q$.

<u>Main Lemma</u> $\Rightarrow$ $R \to \cancel{A} \to C \to \Lambda^{\supset P}$ is resolvable. As $h_A \subset h_C$, this implies $R \to A \to \Lambda \supset P$ resolvable. □

<u>Lemma 12.3</u> (Reduction to local case) Let $R \to A \to \Lambda \supset P$ P min over h_A. $q = \varphi^{-1}(P)$ minimal over $I = \varphi^{-1}(h_A)$. <u>Suppose $\text{ht}\, P = 0$</u>. Then if $R_q \to R_q \otimes_R A \to \Lambda_p^{\supset P\Lambda_P}$ is resolvable, so too is $R \to A \to \Lambda \supset P$.

<u>Pf</u>: Deferred.

<u>Proof</u> of Popescu's Thm in Equicharacteristic-0.

Let $R \to A \to \Lambda \supset P$. $q = \varphi^{-1}(P)$ We can assume $\text{ht}\, q = 0$.

$R_q \to \Lambda_P$ flat $k(q) \to (k(q) \otimes_R \Lambda)_P$ geom. reg.

$$(k(q) \otimes_R \Lambda)_P = \underline{\Lambda_P}$$

<u>Assume</u> $\underline{\mathbb{Q} \subset R_q}$