The Story So Far:

Thm: Let $R \to A \to \Lambda \supset P$ A/R finite-type. $h_A = \sqrt{H_{A/R}\Lambda}$ P minimal over h_A pulls back to $q \subset R$. Assume $R_q \to \Lambda_P$ flat and $k(q) \to \frac{\Lambda_P}{q\Lambda_P}$ geom reg. Then $R \to A \to \Lambda \supset P$ is resolvable (ie $\exists B/R$ f.t st

$R \to A \to B \to \Lambda$ and $h_A \subset h_B \not\subset P$.

Idea: Induct on ht P.

MAIN LEMMA: (11.3-11.4) Let $R \to A \to \Lambda \supset P$ as above Let $a \in R$ map to something in $H_{A/R}$. Suppose

1) $\mathrm{Ann}_R(a) = \mathrm{Ann}_R(a^2)$ & $\mathrm{Ann}_\Lambda(a) = \mathrm{Ann}_\Lambda(a)$.
2) a in A is strictly standard.

Then if $c \geq 8$, $\bar{R} = R/a^c R$, $\bar{A} = A/a^c A$ etc. if $\bar{R} \to \bar{A} \to \bar{\Lambda} \supset \bar{P}$ resolvable, so too is $R \to A \to \Lambda \supset P$.

STANDARDIZATION LEMMA (11.7) Let $R \to A \to \Lambda$ assume $a \in R$ maps to h_A. Then $\exists C/R$ f.t w/ $R \to A \to C \to \Lambda$ w/ a standard in C and $H_{A/R}C \subset H_{C/R}$

Last time we showed that we can reduce to the case that ht $q = 0$. (12.1)

Reduction to the Artinian Local Case

Lemma (12.3): Let $R \to A \to \Lambda \supset P$ and assume that ht $P = 0$. P maps to $q \subset R$. Then if $R_q \to S^{-1}A \to \Lambda_P \supset P\Lambda_P$ is resolvable, so is $R \to A \to \Lambda \supset P$.

Pf: We first show that there is a C/R f.t w/ $R \to A \to C \to \Lambda$ w/ $h_C \not\subset P$ [But NOT necessarily $h_A \subset h_C$].

By resolvability we get $R_q \to S^{-1}A \to B_0 \to \Lambda_P$ w/ $h_{B_0} \not\subset P\Lambda_P \Rightarrow h_{B_0} = (1)$.

By 11.1, we know that we have a smooth B over R_q w/

$R_q \to S^{-1}A \to B \to \Lambda$.

$B = S^{-1}A[X_1 \cdots X_n]/(F_1, \cdots F_m) \longrightarrow \Lambda_P$. $X_i \mapsto \frac{z_i}{s_i} \in \Lambda_P$ can choose common denom.

WLOG $X_i \mapsto \frac{z_i}{s}$ w/ $z_i \in \Lambda$, $s \in \Lambda - P$.

WLOG, coeff of F_i live in A.

If we look at $F_i(x_1 \cdots x_n) \in A[x_1 \cdots x_n, T]$, we can find $N >> 0$ w/

$G_i(\underline{X}, T) = T^N F_i(\frac{X_1}{T}, \frac{X_2}{T}, \cdots \frac{X_n}{T}) \in A[x_1 \cdots x_n, T]$. G_i has NO CONSTANT TERM.

Under $X_i \mapsto z_i$, $T \mapsto s$ we have $G_i \mapsto 0$ in Λ_P. so get $\frac{A[x_1 \cdots x_n, T]}{(G_1 \cdots G_m)} \to \Lambda_P$.

Hence, we have $\forall t \in \Lambda - P$ st $\forall G_i(z_1, \cdots z_n, s) = 0$ in Λ.

Put $t = us \notin P$.

$y_i = uz_i$ $\quad \frac{y_i}{t} = \frac{z_i}{s}$

$$C = A[y_1, \cdots y_n, T] / (G_1, \cdots G_m) \to \Lambda \quad \text{w/} \quad y_i \mapsto y_i, \; T \mapsto t$$

~~[scribble]~~ $G_i(y_1, y_2, \cdots y_n, t) = t^N F_i(\frac{y_1}{t}, \cdots \frac{y_n}{t}) = u^N s^N F_i(\frac{z_i}{s} \cdots) = u^N G_i = (\text{some power of } u) \cdot G(z_1 \cdots z_n, s) = 0$ in Λ.

$$(S^{-1}C)_T = S^{-1}A[y_1 \cdots y_n, T, T^{-1}] / (G_1, \cdots G_m) \underset{y_i = \frac{x_i}{T}}{\cong} \frac{S^{-1}A[x_1 \cdots x_n, T, T^{-1}]}{(F_1 \cdots F_m)}$$

$$\| $$

$$\underset{\text{smooth.}}{R_q} \to B[T, T^{-1}]$$

so ~~[scribble]~~ $T \in H_{S^{-1}C/S^{-1}R}$ $\qquad S = R \setminus q \quad R_q = S^{-1}R$.

$\|$

$T \in S^{-1}(H_{C/R}) \Rightarrow \exists\, r \in S$ st. $rT \in H_{C/R}$.

($S\Lambda \subset \Lambda \setminus P$.)

~~[scribble]~~

~~Both~~ rT maps to $rt \in \Lambda$ but neither r (~~[scribble]~~) nor t is in P.

so $h_C \notin P$.

STEP 2: As $h + P = 0$ Λ_P is artinian so $h_A \Lambda_P$ is nilpotent. $(h_A \Lambda_P)^N = 0$ in Λ_P

$\Rightarrow z \in \Lambda - P$ st $z(h_A)^N = 0$ in Λ.

Put $H_{A/R} = (a_1 \cdots a_r)$ $\quad w_i = $ image of a_i in Λ. $\quad y_i := w_i^N$ $[zw_i = 0]$

Take the C as before so $R \to A \to C \to \Lambda$ w/ $h_C \notin P$.

$$C = A[x_1 \cdots x_n] / (F_1, \cdots F_m) \xrightarrow{\varphi} \Lambda$$

$D := A[\underline{x}, \underline{y}, \underline{T}, \underline{z}] / (F_j - \sum_i y_i T_{ij}, z y_k)$. $y_i \mapsto y_i$ $z \mapsto z$ $T_{ij} \mapsto 0$ $X_i \mapsto \varphi(x_i)$.

$\downarrow$

Λ

Invert one y_k ~~this kills z~~. In D_{y_i}, $z = 0$, ~~so all y's are killed~~.
For each j, we solve $F_j - \sum_i y_i T_{ij}$ for T_{kj}, so that's eliminated.
So D_{y_i} is polynomial over $A[y_k, y_k^{-1}]$. A-smooth.

$D_{a_i y_i}$ —||— $A_a[y_k, y_k^{-1}]$ ← smooth over R.

$a_i y_i \in H_{D|R}$.

$a_i y_i$ maps to $w_i y_i = w_i^{N+1} \in h_D \Rightarrow w_i \in h_D$

But $(w_1 \cdots w_r)$ generate h_A up to radical $\Rightarrow h_A \subset h_D$.

Now $h_C \not\subset P$ so for some $\underline{s \in H_{C|R}}$, ~~s~~ s maps outside of P.

$D_z = A[\underline{x}, \underline{T}, z, z^{-1}] / (F_j) = C[\underline{T}, z, z^{-1}]$ C-smooth.

(inverting z kills all of the y's.)

~~so~~ D_{zs} is R-smooth as $s \in H_{C|R}$.

$zs \in H_{D|R}$ but maps to $\textcircled{z}s \notin P$

↗ not in P by construction.

Pf. of thm in equi-characteristic-0:

$R \to A \to \Lambda \supset P$ P contracts to $q \subset R$ of height 0.

$R_q \to \Lambda_P$ flat $k(q) \to \Lambda_P / q\Lambda_P$ geom regular.

Note that by dim formula, $\dim \Lambda_P = \dim R_q + \dim\left(\frac{\Lambda_P}{qR_P}\right) \Rightarrow \dim \Lambda_P = \dim\left(\frac{\Lambda_P}{q\Lambda_P}\right)$

Since h_A is radical, $h_A \Lambda_P = P\Lambda_P$.
$P \supset h_A$ minimal

Choose $\zeta \in h_A$ such that $\bar{\zeta} \in \frac{\Lambda_P}{q\Lambda_P}$ is a regular parameter.

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Put $R' = R[X] \to A[X] \to \Lambda$, $X \mapsto z$. P maps to $q' \subset R'$ which is $q + (x)$.

(Local flatness criterion) $\Rightarrow$ $R'_{q'} \to \Lambda_p$ is flat. $\Lambda_p / q'\Lambda_p = \frac{\Lambda_p / q\Lambda_p}{(z)}$ regular.

So since $k(q')$ has char 0, $k(q') \to \Lambda_p / q'\Lambda_p$ geom reg.

$\hookrightarrow$ [$R_q \to R'_{q'} \to \Lambda_p$ composite flat, suffices to check mod q get $k[x]_{(x)} \to \frac{\Lambda_p}{q\Lambda_p}$ flat]

~~Now x maps to z ∈ h_A~~

Smoothness preserved under base change $\Rightarrow$ $H_{A|R}\, A[X] \subset H_{A[x]/R[x]}$

so. $h_A \subset h_{A[x]}$

Now $X \mapsto z \in h_A$, so the standardization lemma says for some C

$R' \to A[X] \to C \to \Lambda$ $\quad H_{A[x]/R'} C \subset H_{C|R}$ and image of X standard in C.

Let $N >> 0$ so that X^N satisfies conditions of main lemma.

Put $\overline{R'} := R'/X^{8N}$ $\quad \overline{C} = C/x^{8N}C$ etc.

Since $R'_{q'} \to \Lambda_p$ is flat and x^{8N} is R'_q-regular so too is z^{8N} regular for Λ_p hence, if we ~~look at~~ $ht\, \overline{P} < ht\, P$ for $\overline{P} \subset \overline{\Lambda}$

By the main lemma, we can ~~assume that~~ reduce to $ht\, P = 0$.

By lemma 12.3, we are reduced to proving resolvability for ~~the~~

$R_q \to S^{-1}A \to \Lambda_p$. hence WMA R and Λ are local artinian.

~~Lemma 13.1: Let $R \to \Lambda$ be a flat map of local artinian rings (equichar 0). Assume $m_R \ldots m_R$ geom regular. NB. This is equivalent~~

CLAIM: Let $R \to A \to \Lambda \supset P$ as before w/ R, Λ local artinian. Then it is resolvable.

Rmk: ~~The~~ By assumption $R \to \Lambda$ flat and since $R/m_R \to \Lambda/m_R\Lambda$ geom reg. $\Lambda/m_R\Lambda$ is a field (ie $m_R\Lambda = m_\Lambda$).

It suffices to prove

LEMMA: Let $R \to \Lambda$ be flat local map of artin, equichar 0 rings w/ $m_R\Lambda = m_\Lambda$ then Λ is a direct limit of essentially R-smooth algebras.

⌈ Indeed, this suffices because given $R \to A \to \Lambda$ (A ft), we get

$R \to A \to B \to \Lambda$ (B ess. smooth). $B = S^{-1}$ ~~(...)~~ smooth $/R$.

$$R \to A = R[x_1, \dots x_n]/(F_1 \cdots F_m) \to S^{-1}B \to \Lambda$$

$$A \to B_t \to S^{-1}B$$

only need to invert finitely many things; let t be the product. ⌋

Cor 3.9

Aside: Cohen Structure Theorem: Let k be perfect. Let A be complete local. Let L/k w/ a map $L \to A/m$. Then $\exists$ a lift $L \to A$.

$$\begin{array}{ccc} L & \longrightarrow & A/m \\ \uparrow{\scriptstyle \text{quasi smooth}} & \searrow & \uparrow \\ k & \longrightarrow & A/m^2 \\ & \searrow A \nearrow & \end{array}$$

Inductively get $L \to A/m^i \ \forall i$

$$L \to \varprojlim A/m^i = A.$$

Pf of Lemma: Let $R/m_R = F$, $\Lambda/m_\Lambda = K$. $\Lambda = m_\Lambda \oplus K$ as K-modules.

$\mathbb{Q} \to R$ lifts to $F \to R$. ∴ $F \to R \to \Lambda$ extend. Let $m_R = (\lambda_1, \dots \lambda_r)$.

$$F \to R \to \Lambda \to K, \quad K \dashrightarrow \Lambda$$

~~Pu~~ Put $K[x_1, \dots x_n] \to \Lambda$, $x_i \mapsto \lambda_i$.

Get $\dfrac{K[x_1, \dots x_n]}{(I)} \xrightarrow{\sim} \Lambda$.

$I = (G_1 \cdots G_r)$ WLOG. $X_i^{N_i}$ counted among the G's.

Let $E \subset K$ be a fg field ext'n of F containing all coefficients of the G_i's.

Get $D := \dfrac{E[x_1, \cdots x_n]}{I} \to \Lambda$. D is local artinian

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$$\begin{array}{ccc} R & \longrightarrow & \Lambda \\ & \searrow & \uparrow \\ & & D \end{array}$$

Because image of D contains $\mathbb{F}$ and all generators of m_R [$R = m_R \oplus F$ as F-modules]. R factors thro D.

I claim $R \to D$ is flat.

$$\begin{array}{ccc} E & \xrightarrow{\text{flat}} & K \\ \downarrow & & \downarrow \\ D & \xrightarrow[\text{flat}]{} & K \otimes_E D = \Lambda \end{array}$$

$\Longrightarrow$ $D \to \Lambda$ f. flat.

~~[illegible]~~

$$R \to D \to \Lambda$$

($R \to \Lambda$ flat; $D \to \Lambda$ faithfully flat)

forces $R \to D$ flat.

Given $0 \to M \to N$ in R-mod. get

$$0 \to U \to M \otimes_R D \to N \otimes_R D$$

$$\begin{array}{ccccccc} 0 \to & U \otimes_D \Lambda & \to & M \otimes_R D \otimes_D \Lambda & \to & N \otimes_R D \otimes_D \Lambda \\ & \| & & \| & & \| \\ & 0 & \to & M \otimes_R \Lambda & \to & N \otimes_R \Lambda \end{array}$$

$U = 0$ by faithful flatness.

Since $R \to D$ local.

~~[illegible]~~

If we look at $D/m_R D$ and apply $- \otimes_D \Lambda$, we get $\Lambda/m_R\Lambda = K$

since $D \to \Lambda$ is faith flat, length can only increase $\Rightarrow$ $D/m_R D$ is a field

Thus $R \to D$ w/ $m_R D = m_D$. Take a trans basis for E/F $\{d_1, \dots d_r\}$.

$$(R[x_1 \cdots x_r])_p \to D \qquad x_i \mapsto d_i$$

flat by loc. criterion.

$$(R[x_1 \cdots x_n])_p \to D$$

P = inverse image of m_D.

flat by loc. criterion.

This morphism is clearly unramified: D/PD is a field finite over residue field of $(R[x_1 \cdots x_r])_p$.

$$R \xrightarrow{\text{ess smooth}} R[x_1 \cdots x_r]_p \xrightarrow{\text{étale}} D$$

✓