

The Story So Far:

Thm: Let $R \rightarrow A \rightarrow \Lambda \supset P$ A/R finite-type. $h_A = \sqrt{H_{A/R} \Lambda}$ P minimal over h_A pulls back to $q \subset R$. Assume $R_q \rightarrow \Lambda_P$ flat and $k(q) \rightarrow \frac{\Lambda}{q\Lambda}$ geom. reg.

Then $R \rightarrow A \rightarrow \Lambda \supset P$ is resolvable (ie \exists BIR f.t. st

$$R \rightarrow A \rightarrow B \rightarrow \Lambda \text{ and } h_A \subset h_B \not\subset P.$$

Idea: Induct on $\text{ht } P$.

MAIN LEMMA: (11.3-11.4) Let $R \rightarrow A \rightarrow \Lambda \supset P$ as above. Let $a \in R$ map to something in $H_{A/R}$. ~~Then~~ Suppose

1) $\text{Ann}(a) = \text{Ann}(a^2)$ & $\text{Ann}(A) = \text{Ann}_\Lambda(a)$.

2) a in A is strictly standard.

Then if $e \geq 8$, $\bar{R} = R/a^e R$, $\bar{A} = A/a^e A$ etc. if $\bar{R} \rightarrow \bar{A} \rightarrow \bar{\Lambda} \supset \bar{P}$ resolvable, so too is $R \rightarrow A \rightarrow \Lambda \supset P$.

STANDARDIZATION LEMMA (11.7) Let $R \rightarrow A \rightarrow \Lambda$ assume $a \in R$ maps to h_A .

Then $\exists C/R$ f.t. w/ $R \rightarrow A \rightarrow C \rightarrow \Lambda$ w/ a standard in C and

$$H_{A/R} C \subset H_{C/R}$$

Last time we showed that we can reduce to the case that $\text{ht } q = 0$. (10.1)

Reduction to the Artinian Local Case

Lemma ^(12.3): Let $R \rightarrow A \rightarrow \Lambda \supset P$ and assume that $\text{ht } P = 0$. P maps to $q \subset R$.

Then if $R_q \rightarrow S^{-1}A \rightarrow \Lambda_P \supset P\Lambda_P$ is resolvable, so is $R \rightarrow A \rightarrow \Lambda \supset P$.

pf: We first show that there is a C/R f.t. w/ $R \rightarrow A \rightarrow C \rightarrow \Lambda$ w/ $h_C \not\subset P$ [But NOT necessarily $h_A \subset h_C$].

By resolvability we get $R_q \rightarrow S^{-1}A \rightarrow B_0 \rightarrow \Lambda_P$ w/ $h_{B_0} \not\subset P\Lambda_P \Rightarrow h_{B_0} = (1)$.

By 11.1, we know that we have a smooth B over R_q w/

$$R_q \rightarrow S^{-1}A \rightarrow B \rightarrow \Lambda.$$

$$B = S^{-1}A[x_1, \dots, x_n] / (F_1, \dots, F_m) \rightarrow \Lambda_P. \quad x_i \mapsto \frac{z_i}{s_i} \in \Lambda_P \text{ can choose common denom.}$$

$$\text{wlog } x_i \mapsto \frac{z_i}{s} \text{ w/ } z_i \in \Lambda, s \in \Lambda - P.$$

WLOG, coeff of F_i live in A .

If we look at $F_i(x_1, \dots, x_n) \in A[x_1, \dots, x_n, T]$, we can find $N \gg 0$ w/
 $G_i(x, T) = T^N F_i(\frac{x_1}{T}, \frac{x_2}{T}, \dots, \frac{x_n}{T}) \in A[x_1, \dots, x_n, T]$. G_i has NO
 CONSTANT TERM.

Under $\begin{matrix} x_i \mapsto z_i \\ T \mapsto s \end{matrix}$ we have $G_i \mapsto 0$ so get $A[x_1, \dots, x_n, T] \xrightarrow{(G_1, \dots, G_m)} \Lambda_P$
 in Λ_P .

Hence, we have $u \in \Lambda - P$ st $u G_i(z_1, \dots, z_n, s) = 0$ in Λ .

Put $t = us \notin P$.
 $y_i = uz_i \quad \frac{y_i}{t} = \frac{z_i}{s}$

$C = A[y_1, \dots, y_n, T] / (G_1, \dots, G_m) \xrightarrow{\omega} \Lambda$ w/ $\begin{matrix} y_i \mapsto y_i \\ T \mapsto t \end{matrix}$
 $= t^N F_i(\frac{y_1}{t}, \dots, \frac{y_n}{t}) = u^N s^N F_i(\frac{z_i}{s}, \dots) = u^N G_i$
 $G_i(y_1, y_2, \dots, y_n, t) = (\text{some power of } u) \cdot G(z_1, \dots, z_n, s) = 0$
 in Λ .

$$(S^{-1}C)_T = S^{-1}A[y_1, \dots, y_n, T, T^{-1}] / (G_1, \dots, G_m) \xrightarrow{y_i = \frac{x_i}{T}} S^{-1}A[x_1, \dots, x_n, T, T^{-1}] / (F_1, \dots, F_m)$$

so $T \in H_{S^{-1}C/R}$ $\xrightarrow{R_f \text{ smooth}} B[T, T^{-1}]$
 $S = R1_f \quad R_f = S^{-1}R$

$$T \in S^{-1}(H_{C/R}) \Rightarrow \exists r \in S \text{ st. } rT \in H_{C/R}$$

$S \cap \Lambda - P$

~~Both~~ rT maps to $rt \in \Lambda$ but neither r (~~is in~~)
 nor t is in P .
 so $h_c \notin P$.

STEP 2: As $h_t P = 0$ Λ_P is artinian so $h_A \Lambda_P$ is nilpotent. $(h_A \Lambda_P)^N = 0$
 $\Rightarrow z \in \Lambda - P$ st $z(h_A)^N = 0$ in Λ .

Put $H_{A/R} = (a_1, \dots, a_r)$ $w_i = \text{image of } a_i \text{ in } \Lambda$. $y_i = w_i^N [z w_i = 0]$

Take the C as before so $R \rightarrow A \rightarrow C \rightarrow \Lambda$ w/ $h_c \notin P$.

$$C = A[x_1, \dots, x_n] / (F_1, \dots, F_m) \xrightarrow{\varphi} \Lambda$$

$$D := A[\underline{x}, \underline{y}, \underline{t}, \underline{z}] / (F_j - \sum_i y_i T_{ij}, z y_i) \quad \begin{matrix} y_i \mapsto y_i & z \mapsto z \\ T_{ij} \mapsto 0 & x_i \mapsto \varphi(x_i) \end{matrix} \quad (3)$$

\downarrow
 \wedge

Invert one y_k ~~this kills~~ z . In D_{y_i} , $z = 0$, so ~~all y_i 's are~~ ~~trivial~~.
 For each j , we solve $F_j - \sum_i y_i T_{ij}$ for T_{kj} , so that's eliminated.
 So D_{y_i} is polynomial over $A[y_k, y_k^{-1}]$. A -smooth.

$$D_{y_i} : y_i \dashrightarrow A_A[y_k, y_k^{-1}] \leftarrow \text{smooth over } R.$$

$a_i y_i \in H_{0|R}$.

$a_i y_i$ maps to $w_i y_i = w_i^{n+1} \in h_D \Rightarrow w_i \in h_D$

But (w_1, \dots, w_r) generate h_A up to radical $\Rightarrow h_A \subset h_D$.

Now $h_C \not\subset P$ so for some $s \in H_{C|R}$, ~~set~~ s maps outside of P .

$$D_z = A[\underline{x}, \underline{t}, z, z^{-1}] / (F_j) = C[\underline{t}, z, z^{-1}] \quad C\text{-smooth.}$$

(Inverting z kills all of the y 's.)

So D_{z^s} is R -smooth as $s \in H_{C|R}$.

$z^s \in H_{D|R}$ but maps to $(z^s) \notin P$
 \nearrow
 not in P by construction.

Pf. of thm in ^{epi-}characteristic-0:

$R \rightarrow A \rightarrow \Lambda \supset P$ P contracts to $\mathfrak{q} \subset R$ of height 0.

$R_{\mathfrak{q}} \rightarrow \Lambda_P$ flat $k(\mathfrak{q}) \rightarrow \Lambda_{\mathfrak{q}} / \mathfrak{q} \Lambda_{\mathfrak{q}}$ geom regular.

Note that by dim formula, $\dim \Lambda_P = \dim R_{\mathfrak{q}} + \dim \left(\frac{\Lambda_P}{\mathfrak{q} \Lambda_P} \right) \Rightarrow \dim \Lambda_P = \dim \left(\frac{\Lambda_P}{\mathfrak{q} \Lambda_P} \right)$

Since h_A is radical, $h_A \Lambda_P = P \Lambda_P$.
 $P \supset h_A$ minimal

choose $z \in h_A$ such that $\bar{z} \in \frac{\Lambda_P}{\mathfrak{q} \Lambda_P}$ is a regular parameter.

Put $R[x] \rightarrow A[x] \rightarrow \Lambda$ (4)
 $R' = \quad \quad \quad x \mapsto z.$ P maps to $q' \subset R'$ which is $q + (x).$

Local flatness criterion $\Rightarrow R'_{q'} \rightarrow \Lambda_P$ is flat. $\Lambda_P / q' \Lambda_P = \frac{\Lambda_P / q \Lambda_P}{(z)}$ regular.
 So since $k(q')$ has char 0, $k(q') \rightarrow \Lambda_P / q' \Lambda_P$ geom reg.
 $\rightarrow [R'_q \rightarrow R'_{q'} \rightarrow \Lambda_P$ composite flat, suffices to check mod q set $k(x)_{(x)} \rightarrow \frac{\Lambda_P}{q \Lambda_P}$ flat]

~~Base change to $z \in k_A$~~
 Smoothness preserved under base change $\Rightarrow H_{A|k} A[x] \subset H_{A[x]|R[x]}$
 so $h_A \subset h_{A[x]}$
 Now $x \mapsto z \in h_A$, so the standardization lemma says for some C
 $R' \rightarrow A[x] \rightarrow C \rightarrow \Lambda$ $H_{A[x]|R'} \subset H_{C|k}$ and image of x standard in C .

Let $N \gg 0$ so that x^N satisfies conditions of main lemma.
 Put $\bar{R}' = R' / x^{8N}$ $\bar{C} = C / x^{8N} C$ etc.
 Since $R'_{q'} \rightarrow \Lambda_P$ is flat and x^{8N} is $R'_{q'}$ -regular so too is $\bar{R}'_{q'}$ regular for Λ_P hence, if we ~~took~~ $\text{ht } \bar{P} < \text{ht } P$ for $\bar{P} \subset \bar{\Lambda}$

reduce to

~~By the main lemma, we can assume that~~ $\text{ht } P = 0.$

By lemma 12.3, we are reduced to proving resolvability for ~~the above case~~
 $R_q \rightarrow S^{-1} \rightarrow \Lambda_P$. hence wma R and Λ are local artinian.

Lemma 13.1: Let $R \rightarrow \Lambda$ be a flat map of local artinian rings (equivalently). ~~Then we need to assume~~ Assume $\frac{\Lambda}{m_R \Lambda} \leftarrow \frac{R}{m_R}$ geom regular. This is equivalent

CLAIM: Let $R \rightarrow A \rightarrow \Lambda \supset P$ as before w/ R, Λ local artinian. Then it is resolvable.

Rmk: By assumption $R \rightarrow \Lambda$ flat and since $R/m_R \rightarrow \Lambda/m_R \Lambda$ geom reg. $\Lambda/m_R \Lambda$ is a field (ie $m_R \Lambda = m_\Lambda$).

It suffices to prove

LEMMA: Let $R \rightarrow \Lambda$ be flat local map of artin, equichart 0 rings. w/ $m_R \Lambda = m_\Lambda$ then Λ is a direct limit of ^{essentially} smooth algebras.

Indeed, this suffices because given $R \rightarrow A \rightarrow \Lambda$, we get

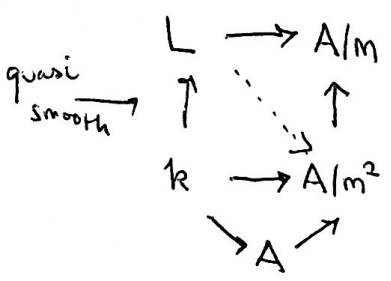
$$R \rightarrow A \xrightarrow{\text{ess. smooth.}} B \rightarrow \Lambda \quad B = \left(\frac{R[x_1, \dots, x_n]}{(f_1, \dots, f_m)} \right) \text{ smooth } / R.$$

$$R \rightarrow A = R[x_1, \dots, x_n] / (f_1, \dots, f_m) \xrightarrow{\text{Car 3.9}} S^{-1} B \rightarrow \Lambda \quad \text{only need to invert finitely many things; let } t \text{ be the product.}$$

$$\downarrow \quad \uparrow$$

$$B_t \rightarrow \Lambda$$

Aside: Cohen Structure Theorem: Let k be perfect. Let A be complete local. Let L/k w/ a map $L \rightarrow A/m$. Then \exists a lift $L \rightarrow A$.



Inductively get $L \rightarrow A/m^i \forall i$
 $L \rightarrow \varinjlim A/m^i = A.$

Pf of Lemma: Let $R/m_R = F$ $\Lambda/m_\Lambda = K$. $\Lambda = m_\Lambda \oplus K$ as K -modules.

$$\begin{array}{ccc}
 \mathbb{Q} \rightarrow R \text{ lifts to } F \rightarrow R & \dots & F \rightarrow R \rightarrow \Lambda \\
 \downarrow & & \downarrow \\
 & & R \\
 & & \downarrow \\
 & & \Lambda \\
 & & \downarrow \\
 & & K
 \end{array}$$

extend. Let $m_R = (\lambda_1, \dots, \lambda_r)$.
 Put $K[x_1, \dots, x_n] \rightarrow \Lambda \quad x_i \mapsto \lambda_i$
 Get $\frac{K[x_1, \dots, x_n]}{(I)} \xrightarrow{\sim} \Lambda.$

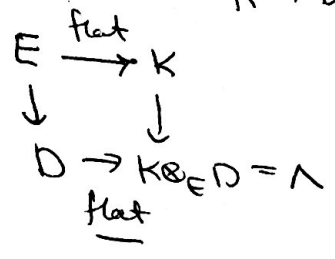
$\subset K$
 Let E be a fg field extn_{OF} containing all coefficients of the G_i 's. $I = (G_1, \dots, G_r)$ wlog. $\mathbb{Q}_i^{N_i}$ counted among the \mathbb{Q}_i 's.

Get $E[x_1, \dots, x_n] \rightarrow \Lambda$. D is local artinian
 $D := \frac{E[x_1, \dots, x_n]}{I}$

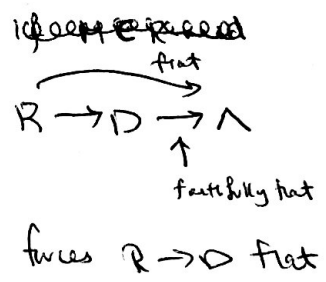
$R \longrightarrow \Lambda$
 $\quad \quad \quad \uparrow$
 $\quad \quad \quad D$

Because image of D contains \mathbb{F} and all generators of m_R [$R = m_R \oplus \mathbb{F}$ at \mathbb{F} -modules].
 R factors thru D .

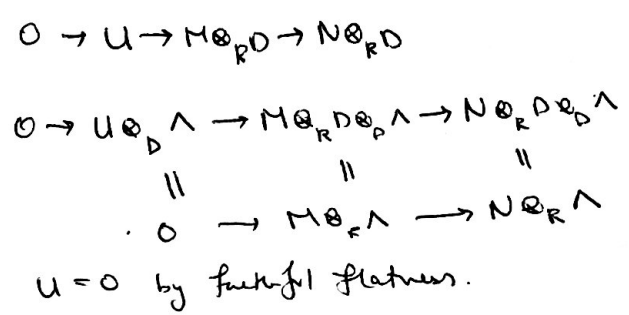
I claim $R \rightarrow D$ is flat.



$\implies D \rightarrow \Lambda$ f. flat.



Given $0 \rightarrow M \rightarrow N$ in R -mod. get



Since $R \rightarrow D$ local.

~~$R \rightarrow E \rightarrow \Lambda$~~

apply

If we look at $D/m_R D$ and $\Lambda \otimes_D \Lambda$, we get $\Lambda/m_R \Lambda = K$
 since $D \rightarrow \Lambda$ is flat, length can only increase $\implies D/m_R D$ is a field

Thus $R \rightarrow D$ w/ $m_R D = m_D$. Take a trans basis for E/\mathbb{F} $\{d_1, \dots, d_r\}$.

$(R[x_1, \dots, x_r])_p \rightarrow D$
 $x_i \mapsto d_i$ flat by loc. criterion.

$(R[x_1, \dots, x_r])_p \rightarrow D$
 $P =$ inverse image of m_D .
 flat by loc. criterion.

This morphism is clearly unramified: D/PD is a field finite over residue field of $(R[x_1, \dots, x_r])_p$.

