SIZES OF SETS

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Definition 0.1. Two sets X and Y *have the same size* if there is a bijection from X to Y.

1. Integers

We will take the following to be axioms for the integers $\mathbb{Z}$.

The integers $\mathbb{Z}$ have two operations $+$ and $\times$, and a relation $<$ that satisfy the following:
Ring axioms

(R1) (Commutativity) For all $a, b \in \mathbb{Z}$, $a + b = b + a$ and $a \times b = b \times a$;
(R2) (Associativity) For all $a, b, c \in \mathbb{Z}$, $(a+b)+c = a+(b+c)$ and $(a\times b)\times c = a\times(b\times c)$;
(R3) (Distributivity) For any $a, b, c \in \mathbb{Z}$, $a \times (b + c) = (a \times b) + (a \times c)$;
(R4) (Identities) There are two distinct elements $0, 1 \in \mathbb{Z}$ such that for any $a \in \mathbb{Z}$ we have $0 + a = a + 0 = a$ and $1 \times a = a \times 1 = a$;
(R5) (Additive inverse) For any $a \in \mathbb{Z}$ there is an element of $\mathbb{Z}$ denoted by $-a$ such that $-a + a = 0$;

Order Axioms

(O1) (Trichotomy) For any elements $a, b \in \mathbb{Z}$ exactly one of the following three statements are true: $a < b$, $a = b$, or $b < a$.
(O2) (Addition) For any $a, b, c \in \mathbb{Z}$ $a < b \Leftrightarrow a + c < b + c$.
(O3) (Multiplication) For any $a, b, c \in \mathbb{Z}$, if $0 < c$ then $a < b \Leftrightarrow ac < bc$ and if $c < 0$ then $a < b \Leftrightarrow bc < ac$.
(O4) (Transitivity) For any $a, b, c \in \mathbb{Z}$, if $a < b$ and $b < c$ then $a < c$.

Well-ordering axiom

We define $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid 0 < n\}$.

(W) (Well-ordering) If A is a non-empty subset of $\mathbb{Z}_+$ then there exists a least element of A — that is, there exists $m \in A$ such that $m \leq a$ for all $a \in A$.

We write $a > b$ to mean $b < a$, and $a \leq b$ (respectively, $a \geq b$) to mean that either $a = b$ or $a < b$ (respectively, $a > b$). For $a \in \mathbb{Z}$ we define $a^0 = 1$, and for $n \in \mathbb{Z}_+$ we define a^n inductively: $a^1 = a$; and $a^{n+1} = a \times a^n$.

Exercise 1.1. Prove that $1 \in \mathbb{Z}_+$.

Exercise 1.2. Prove by induction that if $a \in \mathbb{Z}_+$ then $a^k \in \mathbb{Z}_+$ for all $k \in \mathbb{Z}_+$.

Proposition 1.1. *The least element of $\mathbb{Z}_+$ is 1.*

Proof. By Exercise 1.1 we know that $1 \in \mathbb{Z}_+$. Suppose for contradiction that there exists $a \in \mathbb{Z}_+$ such that $a < 1$. Consider the set $A = \{a, a^2, a^3, \ldots\}$. This is a subset of $\mathbb{Z}_+$ by Exercise 1.2. By the well-ordering axiom A has a least element, which we may write as a^k for some $k \in \mathbb{Z}_+$. Since $a < 1$ and $a^k > 0$ it follows from $(O3)$ that $a^{k+1} < a^k$. This is a contradiction to the fact that a^k is the least element of A. □

Corollary 1.2. *For any integer n, there are no integers a with $n < a < n + 1$.*

Proposition 1.3. *If A is a set of positive integers and if there is an integer N such that $a \leq N$ for all $a \in A$, then there exists a largest element of A — that is, there exists $M \in A$ such that $a \leq M$ for all $a \in A$.*

Proof. Consider the set $U = \{n \in \mathbb{Z}_+ \mid n \geq a \text{ for all } a \in A\}$. The set U is non-empty since $N \in U$. Let M be the least element of U. We must show that $M \in A$. Suppose for a contradiction that $M \notin A$. Then $a \leq M - 1$ for all $a \in A$ by Proposition 1.1. In particular, $M - 1 \in U$. But this contradicts the fact that M is the least element of U, since $M - 1 < M$. □

Definition 1.4. If a set $A \subseteq \mathbb{Z}$ has a largest element we will denote it by $\max A$.

2. Finite Sets

For any positive integer n, let $\mathbb{N}_n = \{m \in \mathbb{Z} \mid 1 \leq m \leq n\}$. Note that $\mathbb{N}_1 \subset \mathbb{N}_2 \subset \ldots$.

Definition 2.1. A set X is said to be *finite* when either

- $x = \emptyset$; or
- there exists a bijection $f : \mathbb{N}_n \to X$ for some positive integer n.

In other words, a set is finite if it is empty or has the same size as $\mathbb{N}_n$ for some $n \in \mathbb{Z}_+$.

Definition 2.2. Let A be a set and $x, y \in A$. We define a function $\tau_{x,y} : A \to A$ by

$$\tau_{x,y}(a) = \begin{cases} y & \text{if } a = x; \\ x & \text{if } a = y; \\ a & \text{otherwise.} \end{cases}$$

Note that $\tau_{x,y}$ is a bijection. If $x = y$ then it is the identity function, and if $x \neq y$ it is called a *transposition*.

Lemma 2.3. *Let X be a set and $x \in X$. Suppose n is a positive integer and $f : X \to \mathbb{N}_n$ is an injection. Then there exists an injection $g : X \to \mathbb{N}_n$ such that $g(x) = n$.*

Proof. We define $g = \tau_{n,f(x)} \circ f$. Then g is an injection since it is the composition of two injections and $g(x) = \tau_{n,f(x)}(f(x)) = n$. □

Proposition 2.4. *Suppose that X is a set, n is a positive integer and $f : X \to \mathbb{N}_n$ is an injection. Then there exists an injection $g : X \to \mathbb{N}_n$ such that $\vec{g}(X) = \mathbb{N}_m$ for some positive integer $m \leq n$.*

Proof. Notice that if $h : X \to \mathbb{N}_n$ is an injection then $\vec{h}(X)$ has a largest element, since $n \geq h(x)$ for all $x \in X$. We consider the set $A = \{\max \vec{h}(X) \mid h : X \to \mathbb{N}_n \text{ is an immersion }\}$. The set A is non-empty since $\max \vec{f}(X)$ is an element of A, and hence A has a smallest element m. Let $g : X \to \mathbb{N}_n$ be an injection such that $\max \vec{g}(X) = m$.

We claim that $\vec{g}(X) = \mathbb{N}_m$. We know that $\vec{g}(X) \subseteq \mathbb{N}_m$, since $g(x) \leq m$ for all $x \in X$. So we must show that $\mathbb{N}_m \subseteq \vec{g}(X)$. We will show this by contradiction. Suppose that $k \in \mathbb{N}_m$ but $k \notin \vec{g}(X)$.

Consider the function $h = \tau_{k,m} \circ g$. This is a function from X to $\mathbb{N}_n$, and it is an injection since it is the composition of an injection and a bijection. In particular, $\max \vec{h}(X) \in A$. The image of h is obtained from the image of g by removing m and adding k. Since $k < m$, we have $\max \vec{h}(X) < m$. But this is a contradiction, since m is the smallest element of A. The contradiction shows that $\vec{g}(X) = \mathbb{N}_m$. □

Proposition 2.5 (Pigeon hole principle)**.** *There is no injection from $\mathbb{N}_{n+1}$ to $\mathbb{N}_n$.*

Proof. We prove this by induction on n. For the base case we must show there is no injection from $\mathbb{N}_2$ to $\mathbb{N}_1$. If $f : \mathbb{N}_2 \to \mathbb{N}_1$ is any function then we have $f(1) = 1$ and $f(2) = 1$, since 1 is the only element of $\mathbb{N}_1$. Thus f is not an injection.

For the induction step assume that there is no injection from $\mathbb{N}_{n+1}$ to $\mathbb{N}_n$. We will prove by contradiction that there is no injection from $\mathbb{N}_{n+2}$ to $\mathbb{N}_{n+1}$. Suppose f were such an injection.

Then by Lemma 2.3 there is an injection $g : \mathbb{N}_{n+2} \to \mathbb{N}_{n+1}$ such that $g(n+2) = n+1$. The restriction $g' = g|_{\mathbb{N}_{n+1}}$ is then an injection (since the restriction of an injection is an injection) from $\mathbb{N}_{n+1}$ to $\mathbb{N}_{n+1}$. But $n+1$ is not in the image of g', since $g(i) \neq n+1$ if $i \neq n+2$. Therefore g' determines an injection $g'' : \mathbb{N}_{n+1} \to \mathbb{N}_n$. (Here the "rule" for g'' is the same as that for g', but the range of g'' is $\mathbb{N}_n$ instead of $\mathbb{N}_{n+1}$.) This contradiction completes the proof of the induction step. □

Corollary 2.6. *There is an injection from* $\mathbb{N}_m$ *to* $\mathbb{N}_n$ *if and only if* $m \leq n$.

Corollary 2.7. *If* X *is a finite set then there is exactly one integer* n *such that there exists a bijection from* $\mathbb{N}_n$ *to* X.

Definition 2.8. If X is a finite set, so there exists a bijection $f : \mathbb{N}_n \to X$, then we write $|X| = n$. (Corollary 2.7 implies that $|X|$ is *well-defined*.)

Exercise 2.1. Suppose $f : X \to Y$ is an injection. Prove that if Y is finite then X is finite and $|X| \leq |Y|$

Exercise 2.2. Suppose $f : X \to Y$ is a surjection. Prove that if X is finite then Y is finite and $|X| \geq |Y|$.

Exercise 2.3. Suppose that X and Y are finite sets of the same size and $f : X \to Y$ is a function. Prove that if f is an injection or f is a surjection then f is a bijection.

3. Infinite Sets

Definition 3.1. A set is said to be *infinite* if it is not finite.

Definition 3.2. A set X is said to be *countable* if

- X is finite; or
- there is a bijection from $\mathbb{Z}_+ to X$.

Proposition 3.3. *If* X *is an infinite set then there is an injection from* $\mathbb{Z}_+$ *to* X.

Proof. We give an inductive definition of a sequence of injections $f_n : \mathbb{N}_n \to X$ with the property that if $m \leq n$ then $f_n|_{\mathbb{N}_m} = f_m$.

Since X is not empty, there exists $x_1 \in X$. Define $f_1(1) = x_1$.

Assume that the injection f_n has been defined for $i \leq n$ and that $f_n|_{\mathbb{N}_m} = f_m$ when $m \leq n$. We claim that f_n is not a surjection, for otherwise $f_n : \mathbb{N}_n \to X$ would be a bijection, contradicting our hypothesis that X is infinite. Since $\vec{f_n}(\mathbb{N}_n) \neq X$, there exists $x_{n+1} \in X - \vec{f_n}(\mathbb{N}_n)$; we define

$$f_{n+1}(i) = \begin{cases} f_n(i) & \text{if } i \leq n; \\ x_{n+1} & \text{if } i = n+1 \end{cases}.$$

For $m \leq n$ we have $f_{n+1}|_{\mathbb{N}_m} = f_n|_{\mathbb{N}_m} = f_m$ as claimed, and f_{n+1} is an injection since f_n is an injection and $f_{n+1}(n+1) \neq f_n(i)$ for any $i < n+1$.

Finally, we define a function $f : \mathbb{Z}_+ \to X$ by $f(n) = f_n(n)$. We must show that f is an injection. Suppose we are given $i, j \in \mathbb{Z}_+$ with $i \neq j$. We must show $f(i) \neq f(j)$. We may assume without loss of generality that $i < j$. Then we have $f(i) = f_i(i) = f_j(i)$ since $f_j|_{\mathbb{N}_i} = f_i$. But $f_j(i) \neq f_j(j)$ since f_j is an injection. □

Theorem 3.4 (Dirichlet)**.** *A set X is infinite if and only if it has a proper subset of the same size as itself.*

Proof. ($\Leftarrow$) Here we prove the contrapositive. If X is finite then X does not have the same size as any proper subset of itself by Proposition 2.6.

($\Rightarrow$) Suppose X is an infinite set. By Proposition 3.3, there is an injection $g : \mathbb{Z}_+ \to X$. We will show that X has the same size as $X - \{g(1)\}$ by constructing a bijection. We define $f : X \to X - \{g(1)\}$ by

$$f(x) = \begin{cases} g(n+1) & \text{if } x = f(n) \text{ for some } n \in \mathbb{Z}_+; \\ x & \text{if } x \notin \vec{g}(\mathbb{Z}_+) \end{cases}.$$

The function f is well-defined, since for any $x \in \vec{g}(\mathbb{Z}_+)$ there is exactly one $n \in \mathbb{Z}_+$ with $g(n) = x$. It is easy to check that f is a bijection. (And you should check this!) □

Exercise 3.1. Modify the proof above to show by induction that if X is infinite and Y is finite then $X \cup Y$ has the same size as X.

Proposition 3.5. *Any subset of $\mathbb{Z}_+$ is countable. In particular, any infinite subset of $\mathbb{Z}_+$ has the same size as $\mathbb{Z}_+$.*

Proof. Suppose that A is a subset of $\mathbb{Z}_+$. If A is finite then it is countable, so we may assume that A is infinite. We will first construct a function $f : \mathbb{Z}_+ \to A$ inductively, and then show that our function is a bijection. To begin the inductive definition, we set $f(1)$ to be the least element of A. For the inductive step, suppose we have defined $f(1), \ldots, f(n)$. Since there is a surjection from $\mathbb{N}_n$ to $\{f(1), \ldots, f(n)\}$, the set $\{f(1), \ldots, f(n)\}$ is a finite subset of A.

Since A is infinite, $A - \{f(1), \ldots, f(n)\}$ is non-empty and hence, by the well-ordering axiom has a least element. We define $f(n+1)$ to be the least element of $A - \{f(1), \ldots, f(n)\}$.

Observe that our inductive definition ensures that $1 \leq i < j \leq n \Rightarrow f(i) \neq f(j)$. Thus f is an injection. We must show that f is a surjection. Note that our inductive definition also guarantees that if $a \in A - \{f(1), \ldots, f(n)\}$ then $f(i) < a$ for $i = 1, \ldots, n$. Assume for contradiction that f is not surjective, so there exists $a \in A - \overrightarrow{f}(\mathbb{Z}_+)$. It follows that $a \in A - \{f(1), \ldots, f(n)\}$ for every positive integer n. In particular, $f(n) < a$ for all n. But this implies that f gives an injection from $\mathbb{Z}_+$ to $\mathbb{N}_n$. This is a contradiction to the pigeon hole principle, since restricting such an injection to the set $\mathbb{N}_{n+1}$ would give an injection from $\mathbb{N}_{n+1}$ to $\mathbb{N}_n$. Thus f must be a bijection. □

Corollary 3.6. *If X is a set and if there exists an injection $f : X \to \mathbb{Z}_+$ then X is countable.*

Proposition 3.7. *If X is a set and if there exists a surjection $f : \mathbb{Z}_+ \to X$ then X is countable.*

Proof. Suppose that $f : \mathbb{Z}_+ \to X$ is a surjection. Since f is a surjection, $\overleftarrow{f}(\{x\})$ is non-empty for every $x \in X$. Thus we may define a function $g : X \to \mathbb{Z}_+$ by letting $g(x)$ be the least element of $\overleftarrow{f}(\{x\})$. The function g is an injection since $x \neq y \Rightarrow \overleftarrow{f}(\{x\}) \cap \overleftarrow{f}(\{x\}) = \emptyset$. Therefore, by Corollary 3.6, X is countable. □

Proposition 3.8. *Suppose that X and Y are disjoint sets. If X is infinite and Y is countable then X has the same size as $X \cup Y$.*

Proof. We consider two cases, according to whether Y is finite or infinite. The case when Y is finite was proved in Problem 3.1. So we may assume that there is a bijection $f : \mathbb{Z}_+ \to Y$. Since X is infinite, we know by Proposition 3.3 that there is a bijection $g : \mathbb{Z}_+ \to X$. Consider the function $h : X \to X \cup Y$ defined as follows:

$$h(x) = \begin{cases} x & \text{if } x \notin \overrightarrow{g}(\mathbb{Z}_+); \\ g(n) & \text{if } x = g(2n) \text{ for some } n \in \mathbb{Z}_+; \textit{ and} \\ h(n) & \text{if } x = g(2n-1) \text{ for some } n \in \mathbb{Z}_+. \end{cases}$$

You should check that the function h is a bijection. (This uses the hypothesis that X and Y are disjoint.) □

4. Big and small infinite sets

Theorem 4.1 (Cantor)**.** *For any set X, there is no surjection from X to $\mathcal{P}(X)$. In particular, X does not have the same size as $\mathcal{P}(X)$.*

Proof. Let $f : X \to \mathcal{P}(X)$ be a function. We must exhibit a subset of X which is not an element of $\vec{f}(X)$. Consider the set $A = \{x \in X \mid x \notin f(x)\}$. Let x be any element of X. We consider two cases: either $x \in A$ or $x \notin A$. If $x \in A$ then $f(x) \neq A$, since $x \notin f(x)$ for all $x \in A$. If $x \notin A$ then the definition of A implies that $x \in f(x)$, so again $f(x) \neq A$. □

Example 4.2. Consider the function $f : \{1, 2, 3, 4\} \to \mathcal{P}(\{1, 2, 3, 4\})$ given by $f(1) = \{1, 2\}$, $f(2) = \{1, 3\}$, $f(3) = \{2, 4\}$, and $f(4) = \{2, 3\}$. Define A as in the proof. Then $A = \{2, 3, 4\}$, which is not in the image of f.

Proposition 4.3. *Let X be a countable set. There is an injection from X to $\mathcal{P}(X)$, but there is no injection from $\mathcal{P}(X)$ to X.*

Proof. We can define an injection from X to $\mathcal{P}(X)$, namely $f(x) = \{x\}$. (This works for any set X, countable or not.)

Next we show that there is no injection from $\mathcal{P}(X)$ to X. This is clear in the case where X is finite, since we showed in our homework that $|\mathcal{P}(X)| = 2^{|X|} > |X|$. If X is infinite, and if $f : \mathcal{P}(X) \to X$ is an injection then we would have a bijection from $\mathcal{P}(X)$ to $\vec{f}(\mathcal{P}(X))$. But there is a bijection between X and any infinite subset of X (since both have bijections to $\mathbb{Z}_+$). Thus by composing we would obtain a bijection from $\mathcal{P}(X)$ to X, contradicting Cantor's Theorem. □

Remark 4.4. In fact, it is true for any set X that there is no injection from $\mathcal{P}(X)$ to X. The proof of this requires the Shröder-Bernstein Theorem, which asserts for two sets X and Y that if there is an injection from X to Y and an injection from Y to X then there exists a bijection between X and Y. This, together with Cantor's theorem, shows that there can be no injection from $\mathcal{P}(X)$ to X.

The goal of the rest of the notes is to show that the set $\mathbb{R}$ of real numbers has the same size as $\mathcal{P}(\mathbb{Z}_+)$. In particular this means that $\mathbb{R}$ is not countable. Since we already saw that the set $\mathbb{Q}$ of rational numbers is countable, this also means that the set of irrational numbers has the same size as $\mathbb{R}$, while $\mathbb{Q}$ is smaller than $\mathbb{R}$, in the sense that there exists an injection from $\mathbb{Q}$ to $\mathbb{R}$ but there does not exist an injection from $\mathbb{R}$ to $\mathbb{Q}$.

Lemma 4.5. *The half-open interval $[0, 1)$ has the same size as $\mathbb{R}$.*

Proof. We will show that $(0, 1)$ has the same size as $\mathbb{R}$. Since $(0, 1)$ is infinite and $[0, 1)$ is the union of $(0, 1)$ with a finite set, namely the set $\{0\}$, they have the same size by Exercise 3.1.

It is easy to show that the function $f(x) = \frac{2x-1}{x-x^2}$ is a bijection from $(0,1)$ to $\mathbb{R}$. (You should show this! It can be done with algebra – calculus is not needed.) $\square$

Definition 4.6. The *binary expansion* of a real number $x \in [0,1)$ is a sequence $(b_1, b_2, \ldots)$ of elements of $\{0,1\}$, where the b_n are defined inductively by:

- If $x < 1/2$ then $b_1 = 0$; otherwise $b_1 = 1$.
- If $x - b_1 2^{-1} - b_2 2^{-2} - \cdots - b_n 2^{-n} < 2^{-(n+1)}$ then $b_{n+1} = 0$; otherwise $b_{n+1} = 1$.

Let $\mathcal{B}$ denote the set of all sequences of elements of $\{0,1\}$. Let $B : [0,1) \to \mathcal{B}$ be the function which maps each number x in $[0,1)$ to its binary expansion.

For $x \in [0,1)$, if $B(x) = (b_1, b_2, \ldots)$ then we set $B_n(x) = b_1 2^{-1} + b_2 2^{-2} + \cdots + b_n 2^{-n}$

Lemma 4.7. *For any $x \in [0,1)$, we have $x \in [B_n(x), B_n(x) + 2^{-n})$.*

Proof. The proof is by induction. For the base case we consider two possibilities: either $x \in [0,1/2)$ or $x \in [1/2,1)$. If $x \in [0,1/2)$ then, by definition, $b_1 = B_1(x) = 0$. Thus $x \in [0,1/2) = [B_1(x), B_1(x) + 2^{-1})$. If $x \in [1/2,1)$ then $b_1 = B_1(x) = 1/2$, so we have $x \in [1/2,1) = [B_1(x), B_1(x) + 2^{-1})$.

For the induction step, assume $x \in [B_n(x), B_n(x) + 2^{-n})$. We again consider two cases, according to whether $x \in [B_n(x), B_n(x) + 2^{-(n+1)})$ or $x \in [B_n(x) + 2^{-(n+1)}, B_n(x) + 2^{-n})$.

In the first case, by definition, $b_{n+1} = 0$ and $B_{n+1}(x) = B_n(x)$. Thus

$$x \in [B_{n+1}(x), B_{n+1}(x) + 2^{-(n+1)}).$$

In the second case, by definition, $b_{n+1} = 1$ and so $B_{n+1}(x) = B_n(x) + 2^{-n+1}$. Thus in the second case we have

$$x \in [B_n(x) + 2^{-(n+1)}, B_n(x) + 2^{-n}) = [B_{n+1}, B_{n+1} + 2^{-(n+1)}).$$

$\square$

Lemma 4.8. *The function B is an injection from $[0,1)$ to $\mathcal{B}$.*

Proof. Suppose that $x, y \in [0,1)$ and $x \neq y$. By the Archimedean Property of $\mathbb{R}$, there is an integer $N > 0$ such that $N > 1/|y-x|$. We proved for homework that $2^N > N$. Thus $|y - x| < 2^{-N}$.

Now assume for contradiction that $B(x) = B(y)$. In particular this implies that $B_N(x) = B_N(y)$. Thus, by Lemma 4.7 we have $x, y \in [B_N(x), B_N(x) + 2^{-N})$, an interval of length 2^{-N}. But this implies that $|x - y| \leq 2^{-n}$, which is a contradiction since we showed above that $|y - x| < 2^{-N}$. $\square$

Definition 4.9. A sequence $(s_n)_{n\in\mathbb{Z}^+}$ of elements of $\{0,1\}$ is *silly* if there exists $N \in \mathbb{Z}_+$ such that $s_n = 1$ for all $n \geq N$. The set of silly sequences will be denoted by $\mathcal{S}$.

Lemma 4.10. *The image* $\vec{B}([0,1))$ *is equal to* $\mathcal{B} - \mathcal{S}$*.*

Lemma 4.11. *The set* $\mathcal{S}$ *is a countable subset of* $\mathcal{B}$*.*

Lemma 4.12. *The set* $\mathcal{B}$ *has the same size as* $\mathcal{P}(\mathbb{Z}_+)$*.*

Theorem 4.13. *The set* $\mathbb{R}$ *has the same size as* $\mathcal{P}(\mathbb{Z}_+)$*. In particular,* $\mathbb{R}$ *is not countable.*

Proof. By Lemma 4.8, $[0,1)$ has the same size as $\vec{B}([0,1))$, which equals $\mathcal{B} - \mathcal{S}$ by Lemma 4.10. But $\mathcal{S}$ is countable by Lemma 4.11 and $\mathcal{B} - \mathcal{S}$ is infinite, since it has the same size as $[0,1)$. Therefore Proposition 3.8 implies that $[0,1)$ has the same size as $\mathcal{B}$. By Lemma 4.12 we know that $\mathcal{B}$ has the same size as $\mathcal{P}(\mathbb{Z}_+)$, and hence $[0,1)$ has the same size as $\mathcal{P}(\mathbb{Z}_+)$. Finally, since $\mathbb{R}$ has the same size as $[0,1)$ by Proposition 4.5, we conclude that $\mathbb{R}$ has the same size as $\mathcal{P}(\mathbb{Z}_+)$. □