(O’Neill 1.1.4) If \(g_1, g_2, g_3,\) and \(h\) are real valued functions on \(\mathbb{R}^3,\) then \(f = h(g_1, g_2, g_3)\) is the function such that \(f(\mathbf{p}) = h(g_1(\mathbf{p}), g_2(\mathbf{p}), g_3(\mathbf{p}))\) for all \(\mathbf{p} \in \mathbb{R}^3.\) Express \(\frac{\partial f}{\partial x}\) in terms of \(x, y\) and \(z\) if \(h = x^2 - yz\) and

(a) \(f = h(x + y, y^2, x + z),\)
(b) \(f = h(e^x, e^{x+y}, e^x),\)
(c) \(f = h(x, -x, x).\)

Solution. Please recall that the chain rule says:

\[
\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}(g_1, g_2, g_3) \frac{\partial g_1}{\partial x} + \frac{\partial h}{\partial y}(g_1, g_2, g_3) \frac{\partial g_2}{\partial x} + \frac{\partial h}{\partial z}(g_1, g_2, g_3) \frac{\partial g_3}{\partial x}
\]

Thus we get

(a) \(\frac{\partial f}{\partial x} = 2(x + y)(1) - (x + z)(0) - (y^2)(1) = 2x + 2y - y^2,\)
(b) \(\frac{\partial f}{\partial x} = 2(e^x)(0) - (e^x)(e^{x+y}) - (e^{x+y})(e^x) = -2e^{2x+y}\)
(c) \(\frac{\partial f}{\partial x} = 2(x)(1) - (x)(-1) - (-x)(1) = 4x.\)

(O’Neill 1.2.4) If \(V = y^2U_1 - x^2U_3\) and \(W = x^2U_1 - zU_2,\) find functions \(f\) and \(g\) such that the vector field \(fV + gW\) can be expressed in terms of \(U_2\) and \(U_3\) only.

Solution. We have

\[fV + gW = (fy^2 + gx^2)U_1 - gzU_2 - fx^2U_3\]

so we only need to choose \(f\) and \(g\) so that \(fy^2 + gx^2 = 0.\) There are many ways to do this, including the “cheating” solution: \(f = g = 0.\) A more interesting solution might be \(f = x^2;\) and \(g = -y^2.\) Then

\[fV + gW = x^2V - y^2W = y^2zU_2 - x^4U_3.\]
(O’Neill 1.2.5) Let $V_1 = U_1 - xU_3$, $V_2 = U_2$, and $V_3 = xU_1 + U_3$.

(a) Prove that the vectors $V_1(p)$, $V_2(p)$, $V_3(p)$ are linearly independent at each point $p \in \mathbb{R}^3$.

(b) Express the vector field $xU_1 + yU_2 + zU_3$ as a linear combination of $V_1$, $V_2$, $V_3$ (where the coefficients are functions on $\mathbb{R}^3$).

Solution.

(a) If $e_1, e_2, \ldots, e_n$ form a basis of a vector space $V$ and if $v_i = \sum a_{ij}e_j$ for $i = 1, \ldots, n$, then the vectors $v_1, \ldots, v_n$ are linearly independent if and only if the matrix $A = (a_{ij})$ has non-zero determinant. In the case at hand, taking $e_i = U_i(p)$ and $v_i = V_i(p)$, we have

$$\det A = \det \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{bmatrix} = x^2 + 1.$$ 

Since $x^2 + 1 > 0$ for all $x$, we conclude that the determinant is non-zero for all points $p$. This shows that $V_1(p)$, $V_2(p)$, $V_3(p)$ are independent for all $p \in \mathbb{R}^3$.

(b) We have $aV_1 + bV_2 + cV_3 = (a + cx)U_1 + bU_2 + (-ax + c)U_3$. Thus we have to solve the equations $a + cx = x; b = y; -ax + c = z$ for the three functions $a, b, c$. Multiplying the first equation by $x$ and adding to the third yields the equation $cx^2 + c = x^2 + z$. Thus we have $c = (x^2 + z)/(x^2 + 1)$. Since $a = x - cx$ we have $a = (x - zx)/(x^2 + 1)$.

Therefore

$$\frac{x - zx}{x^2 + 1} V_1 + y V_2 + \frac{x^2 + z}{x^2 + 1} V_3 = xU_1 + yU_2 + zU_3.$$ 

(O’Neill 1.3.3) Let $V = y^2U_1 - xU_3$ and let $f = xy$, $g = z^3$. Compute the functions (solutions shown below)

(a) $V[f] = y^3$

(b) $V[g] = -3xz^2$

(c) $V[fg] = gV[f] + fV[g] = (y^3)(z^3) + xy(-3xz^2) = y^3z^3 - 3x^2yz^2$

(d) $fV[g] - gV[f] = xy(-3xz^2) - z^3(y^3) = -y^3z^3 - 3x^2yz^2$

(e) $V[f^2 + g^2] = 2fV[f] + 2gV[g] = 2xy^4 - 6xz^5$

(f) $V[V[f]] = 0$
(O’Neill 1.3.4) Prove the identity \( V = \sum V[x_i]U_i \) where \( x_1, x_2 \) and \( x_3 \) are the natural coordinate functions.

**Solution.** The smooth vector field \( V \) on \( \mathbb{R}^3 \) can be uniquely written as \( V = a_1 U_1 + a_2 U_2 + a_3 U_3 \) where \( a_i \) is a smooth function for \( 1 \leq i \leq 3 \). (This is stated as Lemma 2.5 in O’Neill.) Thus to prove the identity it suffices to show that \( a_i = V[x_i] \). By definition, we have \( V[f] = \sum a_j \frac{\partial f}{\partial x_j} \). Since \( \frac{\partial x_i}{\partial x_j} = \delta_{ij} \), we have

\[
V[x_i] = \sum a_j \frac{\partial x_i}{\partial x_j} = \sum a_j \delta_{ij} = a_i.
\]

\( \square \)

(O’Neill 1.3.5) If \( V[f] = W[f] \) for every function \( f \) on \( \mathbb{R}^3 \), prove that \( V = W \).

**Solution.** We can use the identity from the previous problem. Taking \( f \) to be \( x_i \), for each \( i = 1, 2, 3 \), we have \( V[x_i] = W[x_i] \). Thus

\[
V = \sum V[x_i]U_i = \sum W[x_i]U_i = W.
\]

\( \square \)

(O’Neill 1.4.2) Find the unique curve \( \alpha(t) \) such that \( \alpha(0) = (1, 0, 5) \) and \( \alpha'(t) = (t^2, t, e^t) \).

**Solution.** Write \( \alpha(t) = (x(t), y(t), z(t)) \). We must find the functions \( x, y \) and \( z \). By the Fundamental Theorem of Calculus, if \( f(t) \) is continuous then the unique function \( F(t) \) such that \( F(0) = a_0 \) and \( F'(t) = f(t) \) is given by \( F(t) = a_0 + \int_0^t f(s) \, ds \). Thus we find \( x(t) = 1 + t^3/3, y(t) = t^2/2, \) and \( z(t) = 4 + e^t \). This gives

\[
\alpha(t) = (1 + \frac{t^3}{3}, \frac{t^2}{2}, 4 + e^t).
\]

(O’Neill 1.4.4) Reparametrize the curve \( \alpha(t) = (e^t, e^{-t}, \sqrt{2} t) \) using \( h(s) = \log s \) on \( J: s > 0 \). Check the equation in Lemma 4.5 by calculating each side separately.

**Solution.** The reparametrized curve is

\[
\beta(s) = \alpha(h(s)) = (e^{\log s}, e^{-\log s}, \sqrt{2} \log s) = (s, \frac{1}{s}, \sqrt{2} \log s).
\]

Differentiating \( \beta \) yields

\[
\beta'(s) = (1, -\frac{1}{s^2}, \frac{\sqrt{2}}{s}).
\]
On the other hand, we have $\frac{dh}{ds} = 1/s$ and $\alpha'(t) = (e^t, -e^{-t}, \sqrt{2})$. Since $e^h(s) = s$ and $e^{-h(s)} = 1/s$ we get

\[
\frac{dh}{ds}(s)\alpha'(h(s)) = \frac{1}{s}(e^{h(s)}, -e^{-h(s)}, \sqrt{2}) = \frac{1}{s}(s, -\frac{1}{s}, \sqrt{2}) = (1, -\frac{1}{s^2}, \frac{\sqrt{2}}{s}).
\]

(O’Neill 1.4.6-7) Deduce from Lemma 4.6 that in the definition of directional derivative (Def. 3.1) the straight line $t \rightarrow \mathbf{p} + t\mathbf{v}$ can be replaced by any curve $\alpha$ with initial velocity $\mathbf{v}_p$, that is, such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}_p$.

**Solution.** In Definition 3.1 the directional derivative is defined by

\[
\mathbf{v}_p[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))(t=0).
\]

If we define a curve $\lambda$ by $\lambda(t) = \mathbf{p} + t\mathbf{v}$ then the definition may be rewritten as

\[
\mathbf{v}_p[f] = \frac{d}{dt}(f(\lambda))(0)
\]

Now suppose that $\alpha$ is a curve such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}_p$. Since $\lambda'(0) = \mathbf{v}_p$, we have

\[
\frac{d}{dt}(f(\alpha))(0) = \alpha'(0)[f] = \mathbf{v}_p[f] = \lambda'(0)[f] = \frac{d}{dt}(f(\lambda))(0),
\]

where the equalities on the left and right follow from Lemma 4.6.

(a) Show that the curves with coordinate functions

\[
(t, 1 + t^2, t), \quad (\sin t, \cos t, t), \quad (\sinh t, \cosh t, t)
\]

all have the same initial velocity $\mathbf{v}_p$. (b) If $f = x^2 - y^2 + z^2$, compute $\mathbf{v}_p[f]$ by calculating $\frac{d}{dt}(f(\alpha))$ at $t = 0$ using each of the three curves in part (a).

**Solution.**

(a) The velocities are:

\[
(1, 2t, 1), \quad (\cos t, -\sin t, 1), \quad (\cosh t, \sinh t, 1).
\]

Evaluating at $t = 0$ yields $(1, 0, 1)$ in all three cases.

(b) Computing the derivatives, we have

\[
\frac{d}{dt}(f(t, 1 + t^2, t))(t=0) = \frac{d}{dt}(t^2 + (1 + t^2)^2 + t^2)(t=0) = (2t + 2(1 + t^2)(2t) + 2t)|_{t=0} = 0.
\]

\[
\frac{d}{dt}(f(\cos t, -\sin t, 1)) = \frac{d}{dt}(\cos^2 t + \sin^2 t + 1) = \frac{d}{dt}(2) = 0.
\]

\[
\frac{d}{dt}(f(\cosh t, \sinh t, 1))(t=0) = \frac{d}{dt}(\cosh^2 t + \sinh^2 t + 1)|_{t=0} = (4 \cosh t \sinh t)|_{t=0} = 0.
\]