**Problem Set #6 - due Friday, October 7**

For $n \geq 0$ we define the *n-disk* $D^n$ and the *n-sphere* $S^n$ by

- $D^n = \{v \in \mathbb{R}^n : ||v|| \leq 1\}$; and
- $S^n = \{v \in \mathbb{R}^{n+1} : ||v|| = 1\}$

topologized as a subspace of the Euclidean space containing them. (We interpret $\mathbb{R}^0$ to mean the singleton $\{0\}$, where $||0|| = 0$.)

**1.** Let $\sim$ be the equivalence relation on $D^n$, $n > 0$, generated by the relation $x \sim y$ for all $x, y \in S^{n-1}$. (That is, all points of $S^{n-1}$ are identified to one point.) Prove that $D^n/\sim$ is homeomorphic to $S^n$.

**2.** Let $X = \{0,1\} \times \mathbb{R}$, where $\{0,1\}$ has the discrete topology. Let $\sim$ be the equivalence relation on $X$ generated by the relation $(0,x) \sim (1,x)$ for all $x \in (0,1)$.

(a) Show that $X$ is Hausdorff, but $X/\sim$ is not.

(b) Show that there are exactly two pairs of points in $X/\sim$ which do not have the Hausdorff separation property.

**3.** Let $S$ be the strip $\mathbb{R} \times [0,1]$, topologized as a subspace of $\mathbb{R}^2$. Let $\mathbb{Z}$ act on $S$ by $n \cdot (x, y) = (nx, y)$. Let $B = [0,1] \times [0,1] \subset S$.

(a) Show that the quotient space $S/\mathbb{Z}$ is homeomorphic to the annulus $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. Give an explicit quotient map $q : S \twoheadrightarrow A$.

(b) Show that the restriction of $q$ to $B$ is a closed map, and hence a quotient map. (Suggestion: use the theorem from analysis that a bounded sequence in $\mathbb{R}^2$ has a convergent subsequence.)

(c) Give an example of a subspace $Y \subset S$ such that the restriction of $q$ to $Y$ is surjective, but is not a quotient map.

**4.** Let $X = \{0,1\} \times D^2$, where $\{0,1\}$ has the discrete topology.

(a) Let $\sim$ be the equivalence relation generated by the relation $(0,\theta) \sim (1,\theta)$ for all $\theta \in S^1$. Show that $X/\sim$ is homeomorphic to $S^2$.

(b) Let $f : S^1 \to S^1$ be a homeomorphism, and let $\backsim$ be the equivalence relation generated by the relation $(0,\theta) \backsim (1, f(\theta))$ for all $\theta \in S^1$. Show that $X/\sim$ is homeomorphic to $X/\backsim$, and hence that both spaces are homeomorphic to $S^2$.