Math 445 – Fall 2011 Name: ______________________

Problem Set #8 - due Monday, October 31

1. Suppose that X and Y are topological spaces, and that Y is compact. Prove that the projection map $\pi_X : X \times Y \to X$ is a closed map. Give an example to show that this is false if Y is not assumed to be compact.

2. Suppose that $f : X \to Y$ is a function from a topological space X to a compact Hausdorff space Y. Prove that f is continuous if and only if its *graph*

$$\mathcal{G}_f = \{(x, f(x) \mid x \in X\}$$

is a closed set in the product space $X \times Y$.

An action of a group G on a set X is said to be *free* if $g \cdot x \neq x$ for every non-identity element $g \in G$.

Suppose that G acts on a topological space X (so that the functions $x \mapsto g \cdot x$ are required to be homeomorphisms). Then the action is said to be *properly discontinuous* when, for every compact set $K \subset X$, the set $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is finite subset of G.

Recall that an action is said to be *nice* when, for every point $x \in X$, there exists an open neighborhood U of x such that $g \cdot U \cap U = \emptyset$ for all non-identity elements $g \in G$.

3. Prove that if X is locally compact and Hausdorff then every free, properly discontinous action by a group G is nice.

4. Let X be a compact Hausdorff space. For $n \in \mathbb{N}$, suppose that A_n is a closed subset of X with empty interior. Prove that there exists a point of X which is not contained in any A_n.

Let X be a metric space, with metric d. For $x \in X$ and $A \subset X$ define

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Define the ϵ-*neighborhood of* A to the the set $N_\epsilon(A) = \{x \in X \mid d(x, A) < \epsilon\}$.

5.(a) Suppose that $x \in X$ and $A \subset X$ is compact. Prove that there exists $a \in A$ such that $d(x, a) = d(x, A)$. Given an example to show that this fails if A is not assumed to be compact.

(b) Suppose that A and B are disjoint closed sets in X. Prove that if A is compact then A and B have disjoint ϵ-neighborhoods for some $\epsilon > 0$. Give an example to show that this fails if A is not assumed to be compact.