NOTES ON TOPOLOGY

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1. Cells and spaces

We let $D^n = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}$ denote the closed unit disk in $\mathbb{R}^n$ and $S^n = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1\}$ denote its boundary, the unit sphere in $\mathbb{R}^n$. (It is a fact that any homeomorphism from D^n to itself must map S^{n-1} to itself. We have not proven this, but we will assume it for now.)

Definition 1.1. A space homeomorphic to D^n is called an *n-cell*. If C is an n-cell then the *boundary* of C is the image of the $(n-1)$-sphere under a homeomorphism from D^n to C. By the assumption mentioned above, this is a well-defined subset of C, which we denote by ∂C.

Next we will define a class of spaces which includes almost all of the spaces that we will encounter in this course.

Definition 1.2. Suppose that X is a topological space. We will say that Y is obtained from X by *attaching n-cells* if there exist

(1) a family $\mathcal{C}$ of disjoint n-cells C disjoint from X; and
(2) for each $C \in \mathcal{C}$ a map $\alpha_C : \partial C \to X$,

such that Y is homeomorphic to the quotient space $(\bigsqcup \mathcal{C} \sqcup X)/\sim$, where $\sim$ is the equivalence relation generated by the relation $x \sim \alpha_c(x)$ for all $x \in \partial C \in \mathcal{C}$. The map α_C is called the *attaching map* for C.

If Y is obtained from X by attaching n-cells then X is homeomorphic to a subspace of Y. Moreover, for each $C \in \mathcal{C}$, the restriction $q_C : C \to Y$ of the quotient map is called the *characteristic map* of C. We have that $q_C|_{\partial C} = \alpha_C$ and q_C is an embedding of $C - \partial C$ into X. Note, however, that q_C is not necessarily injective on ∂C.

Definition 1.3. A 0*-dimensional CW-complex* is a disjoint union of 0-cells. For $n > 0$, a space X is an n-dimensional CW-complex if X is obtained by attaching n-cells to an $(n-1)$-dimensional CW-complex.

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If X is an n-dimensional CW-complex then there are subspaces

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n)} = X$$

such that each $X^{(k)}$ is a k-dimensional CW-complex and, for each $k > 0$, $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching k-cells. The space $X^{(k)}$ is called the *k-skeleton* (determined by the CW-complex structure) of X. Of course, for most spaces X there are many different ways to describe X as a CW-complex, with different skeleta.

Exercise 1.1. Show that the circle is a CW-complex.

Exercise 1.2. Show that any triangulated 2-manifold is a CW-complex.

2. Connectedness

Definition 2.1. A space X is said to be *path-connected* or 0*-connected* when each pair of points of X can be joined by a path. Equivalently, we could define X to be 0-connected if every map $f : S^0 \doteq \{0, 1\} \to X$ can be extended to a map $\hat{f} : D^1 \doteq [0, 1] \to X$.

A space X is said to be *simply connected*, or 1*-connected* if it is path-connected and every map from the circle S^1 to X extends to a map from D^2 to X.

Continuing inductively for $n > 1$, we say that X is n-connected if it is $(n - 1)$-connected and every map from S^n to X extends to a map from D^{n+1} to X.

We describe here two basic methods for verifying that a space is n-connected.

Proposition 2.2. *A convex subspace of* $\mathbb{R}^n$ *is* m*-connected for all* $m \geq 0$*.*

Proof. Let X be a convex subspace of $\mathbb{R}^n$. Let $f : S^n \to X$ be a map. Choose any point $P \in X$. Every point x in D^n can be written as $x = tv$ for some $v \in S^{n-1}$, and v is uniquely determined unless $x = 0$. (In fact, the disk D^n is the quotient space of $[0, 1] \times S^n$ under the equivalence relation generated by setting $(0, v) \sim (0, v')$ for all $v, v' \in S^n$. We define $\hat{f} : [0, 1] \times S^n$ by $\hat{f}(tv) = (1 - t)P + tf(v)$. Since $\hat{f}$ is constant on equivalence classes, it descends to a map $\hat{f} : D^n \to X$ which is an extension of f. □

Exercise 2.1. Prove that $\mathbb{R}^n$ is m-connected for all $m \geq 0$.

Exercise 2.2. Prove that D^n is m-connected for all $m \geq 0$.

We can generalize the argument above to many more situations.

Definition 2.3. Let X be a topological space and P a point in X. We say that X is *contractible to P* if there exists a continuous function $H : X \times [0, 1] \to X$ such that

- $H(x, 0) = x$ for all $x \in X$;
- $H(x, 1) = P$ for all $x \in X$.

(Here $X \times [0, 1]$ has the product topology, of course.) We call H a *contraction of X to P*.

Exercise 2.3. Prove that if X is contractible to $P \in X$ then X is m-connected for all $m \geq 0$.

Exercise 2.4. Suppose that X is contractible to $P \in X$. Show that X is also contractible to $Q \in X$ for any $Q \in X$.

In view of Exercise 2.4 we will simply say that X *is contractible* if X is contractible to some (and hence any) point $P \in X$.

However, it is important to realize that when a space X is contractible to a point $P \in X$, it may not be possible to find a contraction which keeps the point P fixed.

Exercise 2.5. Give an example of a space X and a point $P \in X$ such that X is contractible to P, but there does not exist a contraction $H : X \times [0, 1] \to X$ with the property that $H(P, t) = P$ for all $t \in [0, 1]$.

Our next criterion for simple connectivity requires a lemma; people who have taken analysis will recognize this as a special case of Lebesgue's covering lemma.

Lemma 2.4. *Let X be a topological space and $\mathcal{U}$ an open cover of X. Let $\sigma : [0, 1] \to X$ be continuous. Then there exist $0 = t_0 < t_1 < \cdots < t_n = 1$ such that, for each $i = 1, \ldots n$, we have $f([t_{i-1}, t_i]) \subset U$ for some $U \in \mathcal{U}$.*

Proof. We will prove this by the method of bisection. A list S of numbers t_i with $0 = t_0 < t_i < \cdots t_n = 1$ determines a *subdivision* of $[0, 1]$ into *subintervals* $[t_0, t_1], \ldots [t_{n-1}, t_n]$. We will say that an interval $[a, b] \subset [0, 1]$ is *good* if $f([a, b]) \subset U$ for some $U \in \mathcal{U}$. With this terminology, our goal is to show that there is a subdivision of $[0, 1]$ such that all of the subintervals are good. We will build this subdivision by successively dividing various subintervals in half.

Observe first that every number $t \in [0, 1]$ is contained in a good interval $[a, b]$ for some $a < t < b$. In fact, if $f(t) \in U$, then t is contained in the interior of an interval that maps into U.

Next consider the following process, starting with the trivial subdivision S_o corresponding to $0 = t_0 < t_1 = 1$. Given a subdivision S_n of $[0, 1]$,

(1) if every subinterval of S_n is good, stop;
(2) otherwise bisect each bad subinterval to form a new subdivision S_{n+1}.

If this process stops after finitely many steps we have produced a subdivision in which every subinterval is good. Thus we may assume, for contradiction, that this process can be repeated arbitrarily many times with a bad interval appearing at each stage. Observe that each bad interval is produced by bisecting a bad interval. Thus we obtain an infinite sequence of bad intervals $I_1 \supset I_2 \supset \cdots$ where the length of I_k is 2^{-k}. By the nested interval property, there exists $t \in \bigcap I_n$. But t is contained in a good interval $[a, b]$ and, since the length of I_k is 2^{-k}, we have $I_k \subset [a, b]$ for all sufficiently large k. This is a contradiction, since it is clear that any subinterval of a good interval is good. □

Exercise 2.6. Generalize Lemma 2.4 to the case of a continous map $H : [0, 1]\times[0, 1] \to X$.

Theorem 2.5. *Suppose that a space X can be written as $X = U \cup V$ where U and V are simply-connected open sets and $U \cap V$ is path-connected. Then X is simply connected.*

Proof. Choose a point $x \in U \cap V$. Let $f : S^1 \to X$ be a map. We will construct an extension of f to a map $\hat{f} : D^2 \to X$.

Note that S^1 can be realized as the quotient space of $[0, 1]$ under the equivalence relation generated by requiring $0 \sim 1$. In fact, the map $q : [0, 1] \to S^1$ defined by $q(t) = (\cos(2\pi t), \sin(2\pi t))$ is a quotient map.

According to Lemma 2.4 there is a subdivision of $[0, 1]$ such that $q \circ f$ maps each subinterval entirely into U or entirely into V. Equivalently, we obtain a subdivision of the circle S^1 into arcs $\theta_1, \ldots, \theta_n \doteq \theta_0$ such that for each $i = 1, \ldots, n$ we either have $f(\theta_i) \subset U$ or $f(\theta_i) \subset V$. Let $p_1, \ldots, p_n \doteq p_0$ be the endpoints of these arcs, so that p_i is an endpoint of both θ_i and θ_{i-1} for each $i = 1, \ldots, n$. Observe that if f maps one arc into U and an adjacent arc into V then the common endpoint of the two arcs must map into $U \cap V$.

Next we consider radial line segments which runs from the origin to p_i. These segments subdivide the disk D^2 into *pizza slices*. A point on the segment from 0 can be uniquely described as tp_i for some $t \in [0, 1]$.

We first extend f to a map $\tilde{f}$ defined on the union of S^1 with these radial segments. To define $\tilde{f}$ on the radial segments, for each $i = 1, \ldots n$ we choose a path ρ_i which runs from the point $x \in U \cap V$ to the point $f(p_i)$. We impose the following requirements on the paths ρ_i:

- if $f(\theta_{i-1}) \subset U$ and $f(\theta_i) \subset U$ then $f(\rho_i) \subset U$;
- if $f(\theta_{i-1}) \subset V$ and $f(\theta_i) \subset V$ then $f(\rho_i) \subset V$;
- otherwise, $f(\rho_i) \subset U \cap V$.

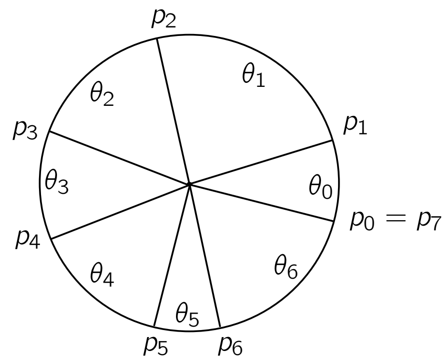


Figure 1.

The last requirement can be satisfied because, in that case, we have $f(p_i) \in U \cap V$ and we have assumed that $U \cap V$ is path-connected.

To finish the argument we observe that the map $\tilde{f}$ has the property that the boundary of each pizza slice is either mapped entirely into U or entirely into V. Since a pizza slice is homeomorphic to D^2, the simple-connectivity of U and V implies that $\tilde{f}$ can be extended over each pizza slice to provide a map $\hat{f} : D^2 \to X$ such that $\hat{f}|_{S^1} = f$, as required. $\square$

Exercise 2.7. Prove that S^n is simply connected for all $n > 1$. Why does your argument fail for $n = 1$?

Exercise 2.8. Prove the following

Theorem 2.6. *Suppose that the space X can be written as $X = U \cup V$ where U and V are n-connected open sets and $U \cap V$ is $(n-1)$-connected. Then X is n-connected.*

3. The Fundamental Group

In this section we will define our most basic algebraic topological invariant – the fundamental group.

We need to establish some terminology first.

Definition 3.1. Suppose that f_0 and f_1 are maps from a space X to a space Y. We say that f_0 is *homotopic* to f_1 if there exists a family of maps $f_t : X \to Y$ for $t \in [0,1]$ such that the function $H : X \times [0,1] \to Y$ given by $H(x,t) = f_t(x)$ is continuous, where $X \times [0,1]$ has the product topology. We may refer to either the map H, or the family f_t as a *homotopy* from f_0 to f_1.

Definition 3.2. Suppose that f_0 and f_1 are maps from a space X to a space Y which agree on a subspace $A \subset X$. We say that f_0 and f_1 are *homotopic rel A* if there exists a homotopy $\{f_t\}_{t\in[0,1]}$ such that $f_t(a) = f_0(a) = f_1(a)$ for all $a \in A$ and all $t \in [0,1]$.

Definition 3.3. By a *path* in a space X we mean a map from $[0,1]$ to X. If $a, b \in X$ are points and $\sigma : [0,1] \to X$ is a path such that $\sigma(0) = a$ and $\sigma)1 = b$ then we may say that σ *runs from* a *to* b, or that a and b are *joined by* σ, or that σ *starts* at a and *ends* at b. By a *loop* based at x we mean a path that starts and ends at the point x.

The set of paths in a space X have what one might call a partial operation denoted $\star$. If σ and τ are two paths such that σ ends at the starting point of τ, then we define

$$\sigma \star \tau(t) = \begin{cases} \sigma(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \tau(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}.$$

Continuity of $\sigma \star \tau$ follows from the condition $\sigma(1) = \tau(0)$. Thus the $\star$ operation concatenates the two paths, if possible, to produce a new path which follows first σ then τ at double speed. While this is a natural operation, its algebraic properties are as weak as possible.

Exercise 3.1. Show that the operation $\star$ is not associative.

To obtain a reasonable algebraic structure we restrict our attention to loops based at a fixed point of X (since the $\star$ operation is defined for any two loops) and we work with equivalence classes of paths under a suitable equivalence relation.

Definition 3.4. We say that two paths σ and τ in a space X are *path homotopic* if they have the same starting and ending points, and are homotopic rel $\{0,1\}$. We will write $\sigma \sim \tau$ when σ and τ are path-homotopic, and we will write $[\sigma]$ for the set of all paths which are path-homotopic to σ.

The next lemma will be used when we define the fundamental group to ensure that the multiplication operation is well-defined.

FOR NOW, REFER TO MASSEY FOR THE DEFINITION OF $\pi_1(X,x)$.

Exercise 3.2. Prove that X is simply connected if and only if X is path-connected and $\pi_1(X,x)$ is trivial for all $x \in X$.

Exercise 3.3. Suppose that X is simply-connected, and that σ and τ are paths in X with $\sigma(0) = \tau(0)$ and $\sigma(1) = \tau(1)$. Prove that σ and τ are path-homotopic.

4. The circle

FOR NOW, REFER TO CLASS NOTES FOR OUR COMPUTATION OF $\pi_1(S^1)$.

This is the key motivating example for the next definitions.

5. Coverings

Definition 5.1. A map $p : \widetilde{X} \to X$ is a *covering* if every point $x \in X$ has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets U_z, for $z \in p^{-1}(x)$, such that, for each $z \in p^{-1}(x)$,

- $z \in U_z$; and
- $p|_{U_z} : U_z \to U$ is a homeomorphism.

The space $\widetilde{X}$ is called the *covering space*, the space X is called the *base space* and the set $p^{-1}(x)$ is called the *fiber* over x.

When an open set U has the property above, it is said to be *evenly covered*, and the open sets U_z are called the *slices* over U.

We will usually want to keep track of a base point when discussing coverings. When a covering is written as a map of pairs, e.g. $p : (\widetilde{X}, \tilde{x}) \to (X, x)$, we mean that $p : \widetilde{X} \to X$ is a covering and $p(\tilde{x}) = x$.

One of the most natural ways for covering to arise is from group actions.

Definition 5.2. An action of a group G on a topological space X is *nice* if it has the following property:

- Every point $x \in X$ has a neighborhood U such that $g \cdot U \cap U = \emptyset$ if $1 \neq g \in G$.

Exercise 5.1. Prove that a free, properly discontinuous action on a locally compact Hausdorff space is nice. (An action of G on X is *free* if no point of X is fixed by any element of G other than the identity. It is *properly discontinuous* if the set $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is finite for every compact set $K \subset X$.

Exercise 5.2. If G acts nicely on $\widetilde{X}$ then the quotient map $q : \widetilde{X} \to X \doteq Q/G$ is a covering.

Exercise 5.3. Prove that the exponential map $t \mapsto e^{2\pi i t}$ is a covering from the real line to the circle.

Exercise 5.4. Prove that the exponential map $z \mapsto e^z$ is a covering from the complex plane $\mathbb{C}$ to $\mathbb{C} - \{0\}$.

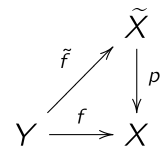
Exercise 5.5. Construct a covering from $\mathbb{R}^2$ to the torus $S^1 \times S^1$.

Exercise 5.6. Construct a covering from the 2-sphere S^2 to the projective plane $\mathbb{R}P^2$.

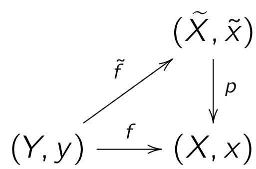
Exercise 5.7. Construct a covering from the hyperbolic plane $\mathbb{H}^2$ to a surface of genus 2.

6. Lifts

Definition 6.1. If $p : \widetilde{X} \to X$ is a covering and if $f : Y \to X$ is a map, we will say that f *lifts* if there exists a map $\tilde{f} : Y \to \widetilde{X}$ such that $p \circ \tilde{f} = f$. The map $\tilde{f}$ is called a *lift* of f. In terms of commutative diagrams, $\tilde{f}$ is a lift of f if the following diagram commutes:



We will also want to keep track of basepoints when discussing lifts. If $f : (Y, y) \to (X, x)$ is a map and if there exists a lift $\tilde{f} : (Y, y) \to (\widetilde{X}, \tilde{x})$ then we will say that f *lifts at* $\tilde{x}$, and $\tilde{f}$ will be called a lift of f at $\tilde{x}$. Thus $\tilde{f}$ is a lift of f at $\tilde{x}$ if the following diagram commutes:



7. Uniqueness of lifts

Lifts do not always exist. However, if a lift does exist, and if the domain is connected and locally connected, then the lift is completely determined by its value at any point of the domain.

Proposition 7.1. *Let $p : \widetilde{X} \to X$ be a covering, and let Y be a connected, locally-connected space. Suppose that $f : Y \to X$ is a map and that $\tilde{f}_1$ and $\tilde{f}_2$ are lifts of f. Then the set $A = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\}$ is either empty, or equal to Y.*

Proof. Since Y is connected, it suffices to show that A and $Y - A$ are both open sets.

Suppose $y \in Y$ and let U be an evenly covered neighborhood of $f(y)$. Let $V \subset f^{-1}(U)$ be a connected neighborhood of y. If U_{z_1} and U_{z_2} are the slices over U that contain $z_1 = \tilde{f}_1(y)$ and $z_2 = \tilde{f}_2(y)$, then U_{z_1} and U_{z_2} are either equal (if and only if $y \in A$), or disjoint. Since V is connected, $\tilde{f}_1(V)$ and $\tilde{f}_2(V)$ are both connected. It follows that $\tilde{f}_1(V) \subset U_{z_1}$ and $\tilde{f}_2(V) \subset U_{z_2}$.

If $y \in A$, we have $U_{z_1} = U_{z_2}$. But each slice over U contains exactly one point in the pre-image of each point of U. Since $\tilde{f}_1(V)$ and $\tilde{f}_2(V)$ are both contained in the same slice this implies that $\tilde{f}_1|_V = \tilde{f}_2|_V$ and hence that $V \subset A$.

If $y \notin A$, U_{z_1} and U_{z_2} are disjoint, and hence $\tilde{f}_1$ and $\tilde{f}_2$ cannot agree at any point of V. Thus we have $V \subset Y - A$. This shows that both A and $Y - A$ are open. □

8. Lifting paths

A crucial property of coverings is that paths, and path homotopies, always lift.

Remark 8.1. As we shall see, it also important that the lift of a loop need not be a loop; that is, $\sigma(0) = \sigma(1)$ does not imply $\widetilde{\sigma}(0) = \widetilde{\sigma}(1)$.

Theorem 8.2. *Let $p : \widetilde{X} \to X$ be a covering. Suppose that $\sigma : [0, 1] \to X$ is a path with $\sigma(0) = x$. Let $\tilde{x}$ be any point in $p^{-1}(x)$. Then there exists a unique path $\widetilde{\sigma} : [0, 1] \to \widetilde{X}$ such that $\widetilde{\sigma}$ is a lift of σ and $\widetilde{\sigma}(0) = \tilde{x}$.*

Proof. Since every point of X has an evenly covered neighborhood, Lemma 2.4 implies that there exists an integer $N > 0$ such that $\sigma([\frac{i}{N}, \frac{i+1}{N}])$ is contained in an evenly covered open set for each $i = 0, \dots, N-1$. It is obvious that the conclusion holds for σ if and only if it holds for any reparametrization of σ. Therefore we may assume that $\sigma = \sigma_1 * \cdots * \sigma_N$ (parentheses omitted since reparametrization is irrelevant), where each path σ_i maps the unit interval into an evenly covered open set. The proof will be by induction on N.

First suppose $N = 1$, so $\sigma_1 = \sigma$. Let U be an evenly covered neighborhood of $\sigma([0, 1])$. Let $q : U \to U_{\tilde{x}}$ be the inverse map of the homeomorphism $p|_{U_{\tilde{x}}} : U_{\tilde{x}} \to U$. Define $\widetilde{\sigma} = q \circ \sigma$.

For the induction step we have $\sigma = (\sigma_1 * \cdots * \sigma_{N-1}) * \sigma_N$ and we know that $\tau = \sigma_1 * \cdots * \sigma_{N-1}$ has a unique lift $\widetilde{\tau}$ at $\tilde{x}$. But we also know by induction that σ_N has a unique lift $\widetilde{\sigma}_N$ at $\widetilde{\tau}(1)$. We define $\widetilde{\sigma} = \widetilde{\tau} * \widetilde{\sigma}_N$. This path is continuous since $\widetilde{\tau}(1) = \widetilde{\sigma}_N(0)$, and it is clearly a lift.

The uniqueness follows immediately from Proposition 7.1. □

Proposition 8.3. *Suppose that $p : \widetilde{X} \to X$ is a covering and that $\sigma : [0, 1] \to X$ and $\tau : [0, 1] \to X$ are paths with $\sigma(0) = \tau(0) = x_0$ and $\sigma(1) = \tau(1) = x_1$. Let $\tilde{x}_0$ be any point in $p^{-1}(x_0)$ and let $\widetilde{\sigma}$ and $\widetilde{\tau}$ be the lifts of σ and τ at $\tilde{x}_0$. If σ and τ are path-homotopic then so are $\widetilde{\sigma}$ and $\widetilde{\tau}$. In particular, $\widetilde{\sigma}(1) = \widetilde{\tau}(1)$.*

Proof. Assume that $H : [0, 1 \times [0, 1] \to X$ is a path-homotopy from σ to τ, so that $H(\{0\} \times [0, 1]) = \{x_0\}$; $H(\{1\} \times [0, 1]) = \{x_1\}$; $H(t, 0) = \sigma(t)$; and $H(t, 1) = \tau(t)$. It suffices to show that H has a lift $\widetilde{H}$ such that $\widetilde{H}(0, 0) = \tilde{x}_0$. For then by uniqueness of path-lifting (including the fact that a lift of a constant path is constant) we must have $\widetilde{H}(\{0\} \times [0, 1]) = \{\tilde{x}_0\}$; $\widetilde{H}(\{1\} \times [0, 1]) = \widetilde{\sigma}(1) = \widetilde{\tau}(1)$; $\widetilde{H}(t, 0) = \widetilde{\sigma}(t)$; and $\widetilde{H}(t, 1) = \widetilde{\tau}(t)$.

By Exercise 2.6, there is an integer $N > 0$ such that each of the squares $S_{i,j} = [\frac{i}{N}, \frac{i+1}{N}] \times [\frac{j}{N}, \frac{j+1}{N}]$ is mapped by H into an evenly covered open set in X, which we regard as a neighborhood of the point $z_{i,j} = H(\frac{i}{N}, \frac{j}{N})$.

Given any point $\tilde{z}_{i,j} \in p^{-1}(z_{i,j})$, it is easy to construct a lift of $H|_{S_{i,j}}$ which maps $(\frac{i}{N}, \frac{j}{N})$ to $\tilde{z}_{i,j}$: if U is an evenly covered open set containing $H(S_{i,j})$ and $\widetilde{U}$ is the slice over U which contains $\tilde{z}_{i,j}$ then we take the lift to be $q \circ H|_{S_{i,j}}$, where q is the inverse map of the homeomorphism $p|_{\widetilde{U}} : \widetilde{U} \to U$.

Define lifts $\widetilde{H}_{i,j}$ of $H|_{S_{i,j}}$ inductively. First, we define $\widetilde{H}_{0,0}$ to be the lift of $H|_{S_{0,0}}$, constructed as above, which sends $(0,0)$ to $\widetilde{\sigma}(0)$. Next we define $\widetilde{H}_{0,1}$ to be the lift of $H|_{s_{1,0}}$ which sends $(\frac{1}{N}, 0)$ to $\widetilde{\sigma}(\frac{1}{N})$. We observe that the restrictions of $\widetilde{H}_{0,0}$ and $\widetilde{H}_{1,0}$ to the segment $S_{0,0} \cap S_{1,0}$ agree, since they are lifts of the same path and they agree at one point. Thus the pasting lemma gives a lift of the restriction of H to $S_{0,0} \cup S_{1,0}$. Continuing by induction we obtain a lift of the restriction of H to the first row $S_{0,0} \cup \cdots \cup S_{N-1,0}$. To continue to the second row, we let $\widetilde{H}_{0,1}$ be the lift of $H|_{S_{0,1}}$ which agrees with $\widetilde{H}_{0,0}$ at the point $(0, \frac{1}{N})$. Again, uniqueness of path lifting guarantees that $\widetilde{H}_{0,0}$ and $\widetilde{H}_{1,0}$ agree on the entire segment $S_{0,0} \cap S_{0,1}$, giving a continuous lift of the restriction of H to the union of the first row and the first square of the second row. We continue across the second row, observing that uniqueness of path-lifting ensures that each new lift agrees with all previous lifts on both segments where the new square meets the union of the previous squares. The proposition follows by induction. □

9. The right action of $\pi_1(X, x)$ on the fiber over x

The results of the preceding section have the following important consequence. Suppose that $p : \widetilde{X} \to X$ is a covering and that $\sigma : [0,1] \to X$ is a loop based at a point $x \in X$. Then

- for each point $z \in p^{-1}(x)$, there is lift $\widetilde{\sigma}$ with $\widetilde{\sigma}(0) = z$;
- $\widetilde{\sigma}$ is a path, but may or may not be a loop;
- The endpoint $\widetilde{\sigma}(1)$ depends only on the path-homotopy class of σ.

Proposition 9.1. *Suppose that $P : \widetilde{X} \to X$ is a covering and that x is a pont of X. There is a unique right action of $\pi_1(X, x)$ on the set $p^{-1}(x)$ such that*

- *if $g \in \pi_1(X, x)$ is represented by the loop σ and if z is a point in $p^{-1}(x)$ then the lift $\widetilde{\sigma}$ of σ at z is a path from z to $z \cdot g$.*

Proof. For $g \in \pi_1(X, x)$ and $z \in p^{-1}(x)$ we would like to define $z \cdot g$ by the following procedure: choose a loop σ representing g; let $\widetilde{\sigma}$ be the lift of σ with $\widetilde{\sigma}(0) = z$; and set $z \cdot g = \widetilde{\sigma}(1)$. Of course we must check that this is well-defined, i.e that it does not depend on the choice of a loop σ representing the element g of $\pi_1(X, x)$. But that is an immediate consequence of the last sentence of Proposition 8.3: if τ is another loop representing g and if $\widetilde{\tau}$ is the lift of τ at z, then $\widetilde{\sigma}(1) = \widetilde{\tau}(1)$.

Next we must verify that this gives a group action, i.e. that $(z \cdot g) \cdot h = z \cdot (gh)$. To describe $(z \cdot g) \cdot h$ we choose loops σ representing h and τ representing g. Let $\widetilde{\tau}$ be the lift of τ at z. Then $\widetilde{\tau}(1) = z \cdot g$. If we let $\widetilde{\sigma}$ be the lift of σ at $\widetilde{\tau}(1)$, then $(z \cdot h) \cdot g = \widetilde{\sigma}(1)$. By construction, $\widetilde{\sigma}(0) = \widetilde{\tau}(1)$. Thus the path product $\widetilde{\tau} * \widetilde{\sigma}$ is defined, and gives a path from z to $(z \cdot g) \cdot h$. The path $\widetilde{\tau} * \widetilde{\sigma}$ is a lift of the path $\tau * \sigma$, which represents the element gh in $\pi_1(X, x)$. Thus we have shown $(z \cdot g) \cdot h = z \cdot (gh)$. ◻

10. Nice actions on simply connected spaces

Here we will show that if a group G acts nicely on a simply connected space $\widetilde{X}$ then the fundamental group of $\widetilde{X}/G$ is isomorphic to G. This gives a powerful method for determining the fundamental group of a space.

Theorem 10.1. *Suppose that the group G acts nicely on a simply connected space $\widetilde{X}$. Let $X = \widetilde{X}/G$. Then $\pi_1(X, x)$ is isomorphic to G for any point $x \in X$.*

Proof. Since G acts nicely, the quotient map from $\widetilde{X}$ to X is a covering. Choose a point $\tilde{x} \in p^{-1}(x)$. Since nice actions are free, and $p^{-1}(x)$ is by definition an orbit, the group G acts freely and transitively on $p^{-1}(x)$. This gives a bijection between elements of G and points of $p^{-1}(x)$: $g \leftrightarrow \tilde{x} \cdot g$.

Next we consider the right action of $\pi_1(X, x)$ on $p^{-1}(x)$. We claim that this action is also free and transitive. To see that it is transitive, let z be any point of $p^{-1}(x)$. Since X is path-connected there is a path $\widetilde{\sigma}$ from $\tilde{x}$ to z. Its projection $\sigma = p \circ \widetilde{\sigma}$ is a loop in X based at x. If γ is the element of $\pi_1(X, x)$ represented by σ then we have $\tilde{x} \cdot \gamma = z$. To see that the action is free, suppose that $z \cdot \gamma = z$. Let σ be a loop in X which represents γ and let $\widetilde{\sigma}$ be the lift of σ at z. Since $z \cdot \gamma = z$, we have $\widetilde{\sigma}(0) = \widetilde{\sigma}(1) = z$. But this implies that γ is in the image of $p_* : \pi_1(\widetilde{X}, z) \to \pi_1(X, x)$, which contains only the identity element since $\pi_1(\widetilde{X}, z)$ is trivial. This shows that the right action is trivial, and gives a bijection between elements of $\pi_1(X, x)$ and points of $p^{-1}(x)$: $\gamma \leftrightarrow \tilde{x} \cdot \gamma$.

Now we consider how the left action of G is related to the right action of $\pi_1(X, x)$. For $g \in G$, $\gamma \in \pi_1(X, x)$ and $z \in p^{-1}(x)$, we claim that $g \cdot (z \cdot \gamma) = (g \cdot z) \cdot \gamma$. To see this, let σ be a loop in X representing γ. If $\widetilde{\sigma}$ is the lift of σ at z then $\widetilde{\sigma}(1) = z \cdot \gamma)$. Consider the path $g \circ \widetilde{\sigma}$ (where we regard g as a function from $\widetilde{X}$ to itself given by $y \mapsto g \cdot y$). We have $g \circ \widetilde{\sigma}(0) = g \cdot z$, and $g \circ \widetilde{\sigma}$ is a lift of σ. Thus $g \circ \widetilde{\sigma}$ is the lift of σ at $g \cdot z$ and we have

$$g \cdot (z \cdot \gamma) = g \cdot \widetilde{\sigma}(1) = g \circ \widetilde{\sigma}(1) = (g \cdot z) \cdot \gamma,$$

as claimed.

If we compose our two bijections we obtain a bijection between the groups G and $\pi_1(X,x)$ given by:

$$g \leftrightarrow \gamma \ \Leftrightarrow \ g \cdot \tilde{x} = \tilde{x} \cdot \gamma.$$

We will check that this is a group homomorphism. Suppose that $g \leftrightarrow \gamma$ and $h \leftrightarrow \eta$. Then

$$(gh) \cdot \tilde{x} = g \cdot (h \cdot \tilde{x}) = g \cdot (\tilde{x} \cdot \eta) = (g \cdot \tilde{x}) \cdot \eta = (\tilde{x} \cdot \gamma) \cdot \eta = \tilde{x} \cdot (\gamma\eta).$$

This shows that our bijection is a group isomorphism, and completes the proof of the Theorem. □

Exercise 10.1. Show that $\pi_1(S^1) \cong C_\infty$, the infinite cyclic group.

Exercise 10.2. Show that $\pi_1(\mathbb{C} - \{0\}) \cong C_\infty$.

Exercise 10.3. Show that $\pi_1(\mathbb{R}P^2) \cong C_2$, the cyclic group of order 2.

Exercise 10.4. Show that the fundamental group of the torus $\mathbb{T}^2$ is a free abelian group of rank 2.

Exercise 10.5. Construct a group of isometries of the hyperbolic plane which is isomorphic to the fundamental group of a sphere with two handles (a surface of genus 2).

11. The general lifting criterion

We are now ready to analyze when a map from a (fairly) general space lifts to a covering.

First, an easy but important observation.

Proposition 11.1. *If $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ is a covering then $p_* : \pi_1(\widetilde{X}, \tilde{x}) \to (X, x)$ is an injection.*

Proof. . Suppose that $\tilde{\sigma}$ is a loop based at $\tilde{x}$ in $\widetilde{X}$ which represents an element g of the kernel of p_*. We will show $g = 1$. Let $\sigma = p \circ \tilde{\sigma}$. Since σ represents $p_*(g)$, we know that there exists a path-homotopy from σ to the constant loop. Since path-homotopies lift, this implies that $\tilde{\sigma}$ is homotopic to the constant loop, and hence $g = 1$. □

Remark 11.2. If $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ is a covering then the group $p_* : \pi_1(\widetilde{X}, \tilde{x}) \to (X, x)$ can be characterized as the subgroup of $\pi_1(X, x)$ consisting of all elements which can be represented by a loop which lifts to a loop at $\tilde{x}$. This is just a tautology, but it is a useful way of thinking about the image group.

Now we can state the general lifting criterion. In view of the remark above, the lifting criterion says that a map $f : (Y, y) \to (X, x)$ lifts at $\tilde{x}$ if and only if f maps each loop in Y based at y to a loop in X based at x which, in turn, lifts to a loop in $\widetilde{X}$ based at $\tilde{x}$.

Theorem 11.3. *Suppose that $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ is a covering, Y is a path-connected, locally path-connected space and $f : (Y, y) \to (X, x)$ is a map. Then f lift to $\tilde{f} : (Y, y) \to (\widetilde{X}, \tilde{x})$ if and only if the subgroup $p_*(\pi_1(Y, y))$ is contained in the subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}))$.*

Proof. If the lift $\tilde{f}$ exists, then by definition we have $p \circ \tilde{f} = f$, so it follows from functoriality of π_1 that the image of f_* is contained in the image of p_*.

Now assume that $p_*(\pi_1(Y, y)) < p_*(\pi_1(\widetilde{X}, \tilde{x}))$. We must construct the map $\tilde{f}$. In view of the fact that paths always lift, the following definition of $\tilde{f}$ is forced upon us. Given $z \in Y$, choose a path τ from y to z. (Such a path exists since Y is path-connected.) Then $\sigma \doteq f \circ \tau$ is a path in X from x to $f(z)$. Let $\widetilde{\sigma}$ be the lift of σ at $\tilde{x}$. Define $\tilde{f}(z) = \widetilde{\sigma}(1)$.

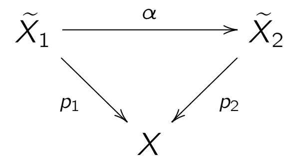
Of course, we must check that this is well-defined, i.e. that if we had chosen a different path from y to z our construction would have produced the same value for $f(z)$. Suppose that τ_1 and τ_2 are paths from y to z. Observe that $\tau_1 * \overline{\tau_2}$ is a loop based at y. If we set $\sigma_1 = p \circ \tau_1$ and $\sigma_2 = p \circ \tau_2$ then $\sigma_1 * \overline{\sigma_2}$ is a loop based at x which represents an element of $p_*(\pi_1(Y, y)$. Since $p_*(\pi_1(Y, y)) < p_*(\pi_1(\widetilde{X}, \tilde{x}))$, the loop $\sigma_1 * \overline{\sigma_2}$ represents an element of $p_*(\pi_1(\widetilde{X}, \tilde{x}))$, which is to say that it lifts to a loop in $\widetilde{X}$ based at $\tilde{x}$. Let $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ denote the lifts of σ_1 and σ_2 at $\tilde{x}$. The statement that $\sigma_1 * \overline{\sigma_2}$ lifts to a loop at $\tilde{x}$ is equivalent to the statement that the lift of $\overline{\sigma_2}$ at $\widetilde{\sigma}_1(1)$ is a path from $\widetilde{\sigma}_1(1)$ to $\tilde{z} = \widetilde{\sigma}_1(0)$. By uniqueness of path lifting, the lift of $\overline{\sigma_2}$ must therfore be equal to $\overline{\widetilde{\sigma}_2}$. Thus $\widetilde{\sigma}_2(1) = \widetilde{\sigma}_1(1)$, which shows that our function $\tilde{f}$ is well-defined.

It remains to check that $\tilde{f}$ is continuous. This is where we will use that Y is locally path-connected. For $z \in Y$, let W be a neighborhood of $\tilde{f}(z)$. We may choose an evenly covered neighborhood U of $f(z)$ such that W contains the slice $U_{f(z)}$ over U which contains $f(z)$. We will show that $\tilde{f}(V) \subset U_{f(z)} \subset W$. Fix a path τ from y to z. For $w \in V$, let ν_w be a path from z to w in V. Since the definition of $\tilde{f}$ does not depend on the choice of path, we may use the path $\tau * \nu_w$ to determine $\tilde{f}(w)$. If we set $\sigma_w = f \circ \nu_w$, and let $\widetilde{\sigma}_w$ be the lift of σ_w at $\tilde{f}(z)$, then according to our definition we have $\tilde{f}(w) = \widetilde{\sigma}_w(1)$. Since the image of σ_w is contained in the evenly covered open neighborhood U of $f(z)$, the lift $\widetilde{\sigma}_w$ has image contained in $U_{f(z)}$ and $\tilde{f}(z) \in U_{f(z)} \subset W$. This shows that $\tilde{f}$ is continuous at z, which was an arbitrary point of Y. □

12. The correspondence theorem

In order to classify coverings, we need to address the question of when two covers are the "same".

Definition 12.1. Suppose that $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are coverings. By a *morphism* from $\widetilde{X}_1$ to $\widetilde{X}_2$ we mean a map $\alpha : X_1 \to X_2$ such that $p_2 \circ \alpha = p_1$. In other words, it is a map which makes the following diagram commute:



The coverings are *isomorphic* if there exist morphisms $\alpha : \widetilde{X}_1 \to \widetilde{X}_2$ and $\beta : \widetilde{X}_2 \to \widetilde{X}_1$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps.

Remark 12.2. It is essential to make the easy obervation that morphisms of covers are, in particular, lifts. One consequence of this is that a morphism is completely determined by the image of a single point. Accordingly, one should keep track of basepoints when analyzing morphisms.

Proposition 12.3. *Suppose $p_1 : (\widetilde{X}_1, \tilde{x}_1) \to (X, x)$ and $p_2 : (\widetilde{X}_2, \tilde{x}_2) \to (X, x)$ are coverings, where X, $\widetilde{X}_1$ and $\widetilde{X}_2$ are connected and locally path-connected. There exists an isomorphism $\alpha : (\widetilde{X}_1, \tilde{x}_1) \to (\widetilde{X}_2, \tilde{x}_2)$ if and only if $p_{1*}(\pi_1(\widetilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \tilde{x}_2))$.*

Proof. This follows immediately from the lifting criterion given in Theorem 11.3. □

We can now prove a converse to Theorem 10.1 under the extra hypothesis that X is connected and locally path-connected.

Theorem 12.4. *Suppose that X is a connected, locally path-connected space and that $p : \widehat{X} \to X$ is a covering with $\widehat{X}$ simply connected. Then $\pi_1(X)$ acts nicely on $\widehat{X}$ with quotient space X.*

Proof. Let us fix basepoints $x \in X$ and $\hat{x} \in p^{-1}(x) \subset \widehat{X}$. Since $\pi_1(\widehat{X}, \hat{x})$ is trivial, Proposition 12.3 implies that the group A of automorphisms of the covering p acts transitively on $p^{-1}(x)$. Proposition 7.1 implies that this action is free, since an isomorphism which fixes a point must be the identity map. As shown in the proof of Theorem 10.1, the right action of $\pi_1(X, x)$ on $p^{-1}(x)$ is also free and transitive. Moreover, the same argument that was used in the proof of Theorem 10.1 gives us an isomorphism between A and $\pi_1(X, x)$, where $\alpha \in A$ corresponds with $\gamma \in \pi_1(X, x)$ if and only if $\alpha \cdot \hat{x} = \hat{x} \cdot \gamma$.

Since the choice of x was arbitrary, we have that the orbits of A are exactly the fibers of p. Thus the only additional thing which must be proven here is that each point z of $\widehat{X}$ has a neighborhood U_z such that $\alpha \cdot U_z \cap U_z = \emptyset$ for every $\alpha \in A - \{1\}$. To see this we use local path-connectedness for the second time. Since X is locally path-connected there exists an evenly covered neighborhood U of $p(z)$ which is path-connected. The slices above U are the path-components of $p^{-1}(U)$, and these must be permuted by each automorphism α. In particular, if we take U_z to be the slice containing z then $\alpha \cdot U_z$ is the slice containing $\alpha(z)$, so we have $\alpha \cdot U_z \cap U_z = \emptyset$ as long as $\alpha \neq 1$. □

We have that each isomorphism class of connected covering spaces of X corresponds to a unique subgroup of the fundamental group of X. It is natural to ask whether every subgroup is associated to a covering. For this to be true, there would have to exist a simply connected covering space, to go with the trivial subgroup. It turns out that the existence of a simply connected covering space is sufficient to give a one-to-one correspondence between subgroups and covering.

Theorem 12.5 (Correspondence Theorem)**.** *Let X be a connected, locally path-connected space and assume that there exists a covering $p : \widehat{X} \to X$ where $\widehat{X}$ is simply connected. Choose a basepoint $x \in X$. Then*

- *for each subgroup $H < \pi_1(X, x)$ there exists a covering $p_H : (\widetilde{X}_H, \tilde{x}_H) \to (X, x)$ such that $\widetilde{X}_H$ is connected and $p_H * (\pi_1(\widetilde{X}_H, \tilde{x}_H)) = H$;*
- *the coverings p_H and p_K are isomorphic if and only if the subgroups H and K are equal;*
- *any covering $p : (\widetilde{X}, \tilde{x}) \to (X, x)$, with $\widetilde{X}$ connected, is isomorphic to p_H for some subgroup $H < \pi_1(X, x)$.*

Proof. All of the conclusions follow from Proposition 12.3 except for the existence of the cover $\widetilde{X}_H$. For this we need to use the simply connected covering space $\widehat{X}$. Since $\pi_1(X, x)$ acts nicely on $\widehat{X}$, the subgroup H also acts nicely. We define $\widetilde{X}_H = \widehat{X}/H$ and we define $\tilde{x}_H$ to be the orbit of $\hat{x}$ under the action of H.

We know from Theorem 10.1 that $\pi_1(tX_h)$ is isomorphic to H, but to show that $p_H * (\pi_1(\widetilde{X}_H, \tilde{x}_H)) = H$ we need to identify exactly how H is acting.

Recall from the proof of Theorem 10.1 that our identification of $\pi_1(X, x)$ with the automorphism group of the covering $p : (\widehat{X}, \hat{x}) \to (X, x)$ pairs an element g of $\pi_1(X, x)$ with an automorphism α when $\alpha(\hat{x}) = \hat{x} \cdot g$. Thus a loop σ in X based at x will lift to a loop in $\widetilde{X}_H$ based at $\tilde{x}_H$ precisely when it lifts to a path in $\widehat{X}$ joining $\hat{x}$ to $\hat{x} \cdot h$ for some $h \in H$. Since the image of p_H* consists of those elements of $\pi_1(X, x)$ which are represented by loops that lift to loops at $\tilde{x}_H$ in $\widetilde{X}_H$, this shows that the image is equal to H. □

13. Automorphisms of coverings

We can now determine the structure of the autormorphism group of an arbitrary connected, locally path-connected covering space.

Theorem 13.1. *Suppose that X is a connected, locally path-connected space and that $p_H : (\widetilde{X}_H, \tilde{x}_H) \to (X, x)$ is the covering corresponding to the subgroup $H < \pi_1(X, x)$. Then the automorphism group of p_H is isomorphic to $N(H)/H$, the quotient of the normalizer of H by H.*

Proof. Let A denote the automorphism group of p_H and choose $\alpha \in A$. According to the easy direction of the lifting criterion, we have $p_{H*}(\pi_1(\widetilde{X}, \alpha(\tilde{x}))) = p_{H*}(\pi_1(\widetilde{X}, \tilde{x})) = H$. On the other hand, if $\widetilde{\sigma}$ is a path from $\tilde{x}$ to $\alpha(\tilde{x})$ then every element of $\pi_1(\widetilde{X}, \alpha(\tilde{x}))$ is represented by a loop of the form $\widetilde{\sigma} * \widetilde{\tau} * \overline{\widetilde{\sigma}}$ where $\widetilde{\tau}$ is a loop based at $\alpha(\tilde{x})$. Setting $\sigma = p_H \circ \widetilde{\sigma}$, and letting g be the element of $\pi_1(X, x)$ represented by σ, this gives $gHg^{-1} = H$. Thus any path from $\tilde{x}$ to $\alpha(\tilde{x})$ projects to a loop in X representing an element of the normalizer of H. Conversely, if g is an element of the normalizer of H, represented by a loop σ, and if $\widetilde{\sigma}$ is the lift of σ at $\tilde{x}$, then $p_{H*}(\pi_1(\widetilde{X}_H, \widetilde{\sigma}(1))) = gHg^{-1} = H$. Thus by the harder direction of the lifting criterion, there exists an automorphism α with $\alpha(\tilde{x}) = \widetilde{\sigma}(1)$.

We may therefore define a surjective function $\phi : N(H) \to A$ as follows. Given $g \in N(A)$, let σ be a loop representing g and let $\widetilde{\sigma}$ be the lift of σ at $\tilde{x}$. Define $\phi(g)$ to be the automorphism which maps $\tilde{x}$ to $\widetilde{\sigma}(1)$. We may express the defining property of ϕ in terms of the path-lifting right action: $\phi(g) = \alpha \Leftrightarrow \alpha(\tilde{x}) = \tilde{x} \cdot g$. It follows formally, much as in the proof of Theorem 10.1 that ϕ is a homomorphism. If $\phi(g) = \alpha$ and $\phi(h) = \beta$ then we have

$$\alpha \circ \beta(\tilde{x}) = \alpha(\tilde{x} \cdot h) = \alpha(\tilde{x}) \cdot h = (\tilde{x} \cdot g) \cdot h = \tilde{x} \cdot (gh),$$

which implies that $\phi(gh)$ maps $\tilde{x}$ to the same point as $\phi(g) \circ \phi(h)$ and, since an automorphism is determined by the image of one point, this implies $\phi(gh) = \phi(h) \circ \phi(h)$.

It remains to show that the kernel of ϕ is H. According to our definition of ϕ, an element of $\pi_1(X, x)$ will be in the kernel of ϕ if it is represented by a loop which lifts to a loop at $\tilde{x}$. According to Remark 11.2 this property characterizes the subgroup H. □

In fact, this theorem could be proved formally, with no reference to path-lifting by identifying the automorphism group with a group A of permutations of $p_H^{-1}(x)$ which acts freely and satisfies $\alpha(\tilde{x} \cdot g) = \alpha(\tilde{x}) \cdot g$ for all $g \in \pi_1(X, x)$. Note that H is the stabilizer of $\tilde{x}$ under the right action.

Given such a permutation α, transitivity of the right action ensures that there exists $g \in \pi_1(X, x)$ such that $\alpha(\tilde{x}) = \tilde{x} \cdot g$. We observe that $g \in N(H)$: if $h \in H$ then

$$\alpha(\tilde{x} \cdot g^{-1}hg)) = \alpha(\tilde{x}) \cdot g^{-1}hg = (\tilde{x} \cdot g) \cdot g^{-1}hg = \tilde{x} \cdot hg = \tilde{x} \cdot g = \alpha(\tilde{x}).$$

Thus $\tilde{x} = \tilde{x} \cdot g^{-1}hg$, which shows $g^{-1}Hg \subset H$. On the other hand,

$$\tilde{x} \cdot (ghg^{-1}) = (\tilde{x} \cdot g) \cdot hg^{-1} = (\alpha(\tilde{x}) \cdot h) \cdot g^{-1} = \alpha(\tilde{x} \cdot h) \cdot g^{-1} = \alpha(\tilde{x}) \cdot g^{-1} = (\tilde{x} \cdot g) \cdot g^{-1} = \tilde{x},$$

which shows $gHg^{-1} \subset H$. Thus $gHg^{-1} = g^{-1}Hg = H$.

Conversely, if $g \in N(H)$, we claim there exists a unique permutation $\alpha \in A$ with $\alpha(\tilde{x}) = \tilde{x} \cdot g$. Every point of $p_H^{-1}(x)$ can be written as $\tilde{x} \cdot k$ for some $k \in \pi_1(X, x)$, so we can attempt to define α by the rule $\alpha(\tilde{x} \cdot k) = \tilde{x} \cdot (gk)$. This uniquely determines α, provided it is well-defined. To show it is well defined we observe that if $k_1 k_2^{-1} \in H$ then

$$\tilde{x} \cdot gk_1k_2^{-1} = \tilde{x} \cdot gk_1k_2^{-1}g^{-1}g = \tilde{x} \cdot g,$$

and hence $\tilde{x} \cdot (gk_1) = \tilde{x} \cdot (gk_2)$.

Now we define $\phi : N(H) \to A$ by letting $\phi(g)$ be the unique element α of A such that $\alpha(\tilde{x}) = \tilde{x} \cdot g$ and proceed as before.

Definition 13.2. Let X and $\widetilde{X}$ be connected, locally path-connected spaces and let $p : \widetilde{X} \to X$ be a covering. We say that p is *normal* if the subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}))$ is normal in $\pi_1(X, x)$ for some (hence every) choice of $x \in X$ and $\tilde{x} \in p^{-1}(x)$.

Observe that if H is a normal subgroup of $\pi_1(X, x)$ then the automorphism group of the covering $p_H : (\widetilde{X}_H, \tilde{x}_H) \to (X, x)$ is isomorphic to the quotient $\pi_1(X, x)/H$.

Proposition 13.3. *A connected covering of a connected, locally path-connected space is normal if and only if its automorphism group acts transitively on each fiber.*

Proof. Suppose X and $\widetilde{X}$ are connected, locally path-connected spaces and $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ is a covering. Set $H = p_*(\pi_1(\widetilde{X}, \tilde{x}))$. Recall, from the proof of Theorem 13.1 that there is an automorphism taking $\tilde{x}$ to $z \in p^{-1}(x)$ if and only if any path from $\tilde{x}$ to z projects to a loop representing an element of $N(H)$.

Assume that H is normal. Given any point $z \in p^{-1}(x)$, let $\tilde{\sigma}$ be a path from $\tilde{x}$ to z. The projection $\sigma = p \circ \tilde{\sigma}$ obviously represents an element of $N(H)$, since $N(H) = \pi_1(X, x)$. Thus there exists an automorphism mapping $\tilde{x}$ to z. This implies that the automorphism group acts transitively

Conversely, suppose that the automorphism group acts transitively on $p^{-1}(x)$. We will show that $N(H) = \pi_1(X, x)$, hence H is normal. Choose any element $g \in \pi_1(X, x)$. Let σ be a loop representing g and let $\tilde{\sigma}$ be its lift at $\tilde{x}$. Consider the point $z = \tilde{\sigma}(1)$. By transitivity, there exists an automorphism taking $\tilde{x}$ to z. Thus $g \in N(H)$, as required. $\square$

14. Gluing spaces together: Van Kampen's theorem

Stay tuned!