A TALE OF TWO FUNCTORS

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1. Hom and Tensor

It was the best of times, it was the worst of times, it was the age of covariance, it was the age of contravariance, it was the epoch of homology, it was the epoch of cohomology, it was the season of Ext, it was the season of Tor, it was the spring of short exact sequences, it was the winter of long exact sequences, we had right exactness, we had left exactness, our arrows were all going in one direction, our arrows were all reversed – in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, as the age of homological algebra.

This is the story of two functors, Hom and Tensor. Hom produces cohomology groups while Tensor produces fancy homology groups in which homology classes are represented by chains whose coefficients may not be integers. Hom is easy to describe, but is a bit kinky due to its trait of reversing arrows. It is a contravariant functor. Tensor, on the other hand, is harder to define but it keeps arrows pointing in the same direction; it is covariant.

First we will introduce Hom. Let R be a commutative ring with 1 and A an R-module. The module A determines a functor from the category of R-modules to itself which acts as follows:

- $M \rightsquigarrow \mathrm{Hom}_R(M, A)$;
- $f : M \to N \rightsquigarrow f^* : \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(M, A)$, where $f^*(g) = g \circ f$.

We will denote this functor by $\mathrm{Hom}_R(\_\,, A)$. The notation f^* for the image of a morphism f under $\mathrm{Hom}_R(\_\,, A)$ could be problematic since it contains no reference to the module A. Hopefully it will be clear from the context which module A is being used to define the functor whenever we use this notation. We remark that the commutativity of R is required in order for $\mathrm{Hom}_R(M, A)$ to have the structure of an R-module. If R were not commutative, $\mathrm{Hom}_R(M, A)$ would be nothing more than a mere abelian group.

For those who may not have seen the definition of the tensor product lately, or ever, we give a very brief review before introducing Tensor. We will only consider modules over commutative rings, and only mention the most basic properties. An algebra class will have much more to say about this topic.

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Definition 1.1. Let A and B be R-modules. Let F be the free R-module with basis indexed by the set $A \times B$. Denote the basis element indexed by (a, b) as $[a, b]$. The module $A \otimes_R B$ is defined to be the quotient of F by the submodule generated by all elements of F of the following forms:

- $[r_1\mathbf{a}_1 + r_2\mathbf{a}_2, \mathbf{b}] - r_1[\mathbf{a}_1, \mathbf{b}] - r_2[\mathbf{a}_2, \mathbf{b}]$ for all $r_1, r_2 \in R$, $\mathbf{a}_1, \mathbf{a}_2 \in A$ and $\mathbf{b} \in B$.
- $[\mathbf{a}, r_1\mathbf{b}_1 + r_2\mathbf{b}_2] - r_1[\mathbf{a}, \mathbf{b}_1] - r_2[\mathbf{a}, \mathbf{b}_2]$ for all $r_1, r_2 \in R$, $\mathbf{a} \in A$ and $\mathbf{b}1, \mathbf{b}_2 \in B$.

The coset of the relation submodule which contains $[a, b]$ will be denoted $a \otimes b$.

It is clear from the definition that the following relations hold in $A \otimes_R B$:

- $(r_1\mathbf{a}_1 + r_2\mathbf{a}_2) \otimes \mathbf{b} = r_1(\mathbf{a}_1 \otimes \mathbf{b}) + r_2(\mathbf{a}_2 \otimes \mathbf{b})$
- $\mathbf{a} \otimes (r_1\mathbf{b}_1 + r_2\mathbf{b}_2) = r_1(\mathbf{a} \otimes \mathbf{b}_1) + r_2(\mathbf{a} \otimes \mathbf{b}_2)$

While a module defined this way is a fairly incomprehensible object, there is one thing (perhaps the only thing) that is readily apparent from the definition, namely that $A \otimes_R B$ has a universal property. If A, B and C are R-modules, we will say that a homomorphism $\phi : A \times B \to C$ is *bilinear* if

- $\phi(r_1\mathbf{a}_1 + r_2\mathbf{a}_2, \mathbf{b}) = r_1\phi(\mathbf{a}_1, \mathbf{b}) + r_1\phi(\mathbf{a}_2, \mathbf{b})$; and
- $\phi(\mathbf{a}, r_1\mathbf{b}_1 + r_2\mathbf{b}_2) = r_1\phi(\mathbf{a}, \mathbf{b}_1) + r_2\phi(\mathbf{a}, \mathbf{b}_2)$

The prototypical example of a bilinear homomorphism is a product of two homomorphisms to R. If $f : A \to R$ and $g : B \to R$ are homomorphisms, then $(a, b) \mapsto f(a)g(b)$ is a bilinear homomorphism from $A \times B \to R$. More generally, whenever the target C has the structure of an R-algebra one can construct bilinear homomorphisms in this way. But they arise in other ways too. In particular, there is a well-defined bilinear homomorphism from $A \times B$ to $A \otimes_R B$ given by $(a, b) \mapsto a \otimes b$.

Proposition 1.2. *The R-module $A \otimes_R B$ has the following universal property. Given any bilinear homomorphism $\varphi : A \times B \to C$ there exists a unique R-module homomorphism $\Phi : A \otimes_R B \to C$ making the following diagram commute:*

$$\begin{tikzcd}
A \times B \arrow[r, "(a,b)\mapsto a\otimes b"] \arrow[dr, "\varphi"] & A \otimes_R B \arrow[d, "\Phi"] \\
 & C
\end{tikzcd}$$

Proof. Let F be the free module with basis indexed by $A \times B$, as in the definition of $A \otimes_R B$. Define a homomorphism from F to C by sending the basis element $[a, b]$ to $\varphi(a, b)$. Since this homomorphism sends relators to 0, it induces a homomorphism Φ which makes the diagram commute. Any homomorphism Φ making the diagram commute would be induced by this same homomorphism from F to C. Thus Φ is unique. $\square$

Example 1.3. Let's take $R = \mathbb{Z}$, so we are considering abelian groups, and compute a few examples. Here, and whenever we work with $\mathbb{Z}$-modules, we will drop the subscript and use $\otimes$ to mean $\otimes_{\mathbb{Z}}$.

If G is an abelian group and F is a free abelian group with basis $\mathcal{B}$ then $\mathrm{Hom}(F, G)$ is isomorphic to the direct product $\prod_{\mathcal{B}} G$ of copies of G indexed by $\mathcal{B}$. This is just the statement that homomorphisms from F to G are in one to one correspondence with *functions* from $\mathcal{B}$ to G, together with the observation that the correspondence is a homomorphism.

In the same setting, we find that $F \otimes G$ is isomorphic to the direct sum $\bigoplus_{\mathcal{B}} G$ of copies of G indexed by the basis $\mathcal{B}$. To see this, recall that the direct sum $\bigoplus_{\mathcal{B}} G$ is characterized in terms of the inclusion homomorphism i_b for $b \in \mathcal{B}$. We must check that $F \otimes G$ has a family of inclusion homomorphisms with the same universal property. For each $b \in \mathcal{B}$ we take $i_b(g) = b \otimes g \in F \otimes B$. Given any family f_b of homomorphisms from G to an abelian group A we must construct a homomorphism from $F \otimes B$ to A which sends $b \otimes g$ to $f_b(g)$. For this it suffices to check the bilinearity of the homomorphism from $F \times G$ to A given, for an arbitrary finite subset B of $\mathcal{B}$, by $(\sum_{b \in B} n_b b, g) \mapsto \sum_{b \in B} n_b f_b(g)$.

Finally, given two integers m and n we figure out the structure of the abelian group $\mathbb{Z}_m \otimes \mathbb{Z}_n$; we will show that it is isomorphic to $\mathbb{Z}_r$ where r is the gcd of m and n. First observe that $\mathbb{Z}_m \otimes \mathbb{Z}_n$ is a cyclic group generated by the element $1 \otimes 1$. In fact, we have $a \otimes b = ab(1 \otimes 1)$, so every element of $\mathbb{Z}_m \otimes \mathbb{Z}_n$ is a sum of multiples of $1 \otimes 1$, and hence is itself a multiple of $1 \otimes 1$. Next observe that the generator $1 \otimes 1$ has order dividing r. To see this write $r = pm + qn$ and compute that

$$r(1 \otimes 1) = pm(1 \otimes 1) + qn(1 \otimes 1) = pm \otimes 1 + 1 \otimes qn = 0$$

since $pm = 0$ in $\mathbb{Z}_m$ and $qn = 0$ in $\mathbb{Z}_n$. Thus the order of the cyclic group $\mathbb{Z}_m \otimes \mathbb{Z}_n$ divides r. On the other hand, the universal property implies that $\mathbb{Z}_m \otimes \mathbb{Z}_n$ maps onto $\mathbb{Z}_r$, since there is a surjective bilinear map from $\mathbb{Z}_m \times \mathbb{Z}_n$ to $\mathbb{Z}_r$ given by $(a, b) \mapsto ab \mod r$.

Now we are ready to introduce Tensor. Again, we assume that R is a commutative ring with 1 and that A is an R-module. Then A determines a functor, denoted $\_ \otimes_R A$, from the category of R-modules to itself which is given by:

- $M \rightsquigarrow M \otimes_R A$; and
- $f : M \to N \rightsquigarrow f \otimes \mathrm{id} : M \otimes_R A \to N \otimes_R A$.

Here $f \otimes \mathrm{id}$ is determined by the bilinear homomorphism $(m, a) \mapsto f(m) \otimes a$.

It is immediate from functoriality that $\_ \otimes_R A$ carries chain complexes to chain complexes and that $\mathrm{Hom}_R(\_, A)$ carries chain complexes to cochain complexes (i.e. differential graded R-modules with differential of graded degree 1, rather than -1).

It therefore makes sense to take a chain complex C, apply one or the other of these functors, and consider the (co)homology groups of the resulting (co)chain complex. The cohomology groups obtained from the cochain complex $\mathrm{Hom}(C, G)$ are called *cohomology groups with G coefficients* and are denoted $H^n(C; G)$. The homology groups of $C \otimes G$ are *homology groups with G coefficients* and are denoted $H_n(C; G)$.

A key feature of the functors $\mathrm{Hom}_R(\_\,, A)$ and $\_ \otimes_R A$ is that they are *additive*, meaning that they respect the R-module structure on $\mathrm{Hom}_R(M, N)$. Thus if $f : M \to N$ and $g : M \to N$ are R-module homomorphisms and if $\mathcal{F}$ denotes either of the functors, then $\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)$. A consequence of this is that these functors carry (co)chain homotopies to (co)chain homotopies. (Recall that the definition of chain homotopy requires addition of homomorphisms.)

2. Ext **and** Tor

An interesting aspect of Hom and Tensor is that they are not exact functors – they do not necessarily carry short exact sequences to short exact sequences. But they almost do.

Proposition 2.1. *Let R be a commutative ring with 1 and let*

$$0 \longrightarrow K \xrightarrow{i} E \xrightarrow{p} Q \longrightarrow 0$$

be a short exact sequence of R-modules. Then for any R-module A the sequence

$$\mathrm{Hom}_R(K, A) \xleftarrow{i^*} \mathrm{Hom}_R(E, A) \xleftarrow{p^*} \mathrm{Hom}_R(Q, A) \longleftarrow 0$$

is exact. (But the map i^* may not be surjective.)

Proof. It follows from the surjectivity of p that $f \circ p = 0 \Rightarrow f = 0$, so p^* is injective. The equality $\ker(i^*) = \mathrm{im}(p^*)$ is equivalent to the statement that a homomorphism $E \to A$ restricts to the zero homomorphism on K if and only if it induces a homomorphism $Q \to A$. $\square$

The analogous statement for $\_ \otimes_R$ takes a bit more work.

Proposition 2.2. *Let R be a commutative ring with 1 and let*

$$0 \longrightarrow K \xrightarrow{i} E \xrightarrow{p} Q \longrightarrow 0$$

be a short exact sequence of R-modules. Then for any R-module A the sequence

$$K \otimes_R A \xrightarrow{i \otimes \mathrm{id}} E \otimes_R A \xrightarrow{p \otimes \mathrm{id}} Q \otimes_R A \longrightarrow 0$$

is exact. (But the map $i \otimes \mathrm{id}$ may not be injective.)

Proof. It follows from functoriality that $(p \otimes \mathrm{id}) \circ (i \otimes \mathrm{id}) = 0$, and it follows from the fact that $Q \otimes_R A$ is generated by elements of the form $q \otimes a$ that $p \otimes \mathrm{id}$ is surjective. To show that $\mathrm{im}(i \otimes \mathrm{id}) = \ker(p \otimes \mathrm{id})$ it suffices to show that the induced homomorphism

$$\hat{p} : (E \otimes_R A)/im(i \otimes \mathrm{id}) \to Q \otimes_R A$$

is an isomorphism, which we will do by exhibiting an inverse. The construction of the inverse uses the universal property of $Q \otimes_R A$. Suppose (q, a) is an element of $Q \times A$. Choose an element $e \in M$ with $p(e) = q$. Define $f(q, a) = [e \otimes a]$, where the square brackets denotes a coset of $\mathrm{im}(i \otimes \mathrm{id})$. The definition does not depend on the choice of e because any two choices for e will differ by an element of $\mathrm{im}(i)$. It is straightforward to check that f is a bilinear homomorphism, and hence induces a homomorphism $Q \otimes_R A \to (E \otimes_R A)/im(i \otimes \mathrm{id})$. It is also straightforward to check that this homomorphism is an inverse to $\hat{p}$. $\square$

Example 2.3. It is easy to give an example to show that the homomorphism $i \otimes \mathrm{id}$ in the conclusion of Proposition 2.2 need not be injective. Consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. If we tensor with $\mathbb{Z}_2$ then we obtain the exact sequence $\mathbb{Z}_2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$ where the arrow on the left is 0 and the middle arrow is the identity. Moreover, the same example shows that the homomorphism i^* in the conclusion of Proposition 2.1 need not be surjective. If we apply the functor $\mathrm{Hom}(\_\,, \mathbb{Z})$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ we obtain the exact sequence $\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}_2 \leftarrow 0$ where the arrow on the left is multiplication by 2 and the middle arrow is 0.

It turns out that there are homological invariants of A, named Ext and Tor which are constructed as (co)homology groups of certain (co)chain complexes and which can be viewed as measuring the extent to which $\_ \otimes_R A$ and $\mathrm{Hom}_R(\_\,, A)$ fail to be exact. We need a definition to construct these invariants.

Definition 2.4. Let R be a commutative ring with 1, and let M be an R-module. A *free resolution* of M is an exact sequence of R-modules

$$\cdots \to F_n \to F_{n-1} \to \cdots F_1 \to F_0 \to M \to 0$$

such that each F_n is free. Equivalently (and more usefully) a free resolution of M is a chain complex F such that

- Each F_n is a free R-module, and the differentials are R-module homomorphisms;
- $H_0(F) = M$ and $H_n(F) = 0$ for $n > 0$.

Example 2.5. The most basic example of a free resolution, and the one which we are primarily interested in, is a presentation of an abelian group A:

$$0 \to K \to F_0 \to A \to 0.$$

Here $R = \mathbb{Z}$ and $F_n = 0$ for $n > 1$. To construct a presentation of the abelian group A one chooses a set of generators. The free $\mathbb{Z}$-module F_0, with basis indexed by the chosen generators, maps onto A. The relation subgroup is the kernel K of this homomorphism. Since submodules of a free $\mathbb{Z}$-module are free, the short exact sequence is a special type of free resolution.

When the abelian group A above is replaced by an R-module M over a general commutative ring R, it is still possible to find a free R-module F_0 with a surjective homomorphism to M. The difference is that the kernel K may not be free in this situation. However, if K is not free then one can find another free module F_1 which has a surjective homomorphism to K. If one repeats this process ad infinitum, the result is a free resolution of M.

Definition 2.6. Suppose E is a free resolution of an R-module M and F is a free resolution of an R-module N. Let $\varphi : M \to N$ be a homomorphism. We will say that a chain map $\Phi : E \to F$ *extends* φ if the induced homomorphism $\Phi_* : H_0(E) \to H_0(F)$ agrees with φ. (Here we are identifying $H_0(E)$ with M and $H_0(F)$ with N.)

Lemma 2.7. *Let F be a free R-module and let $q : M \to N$ be a homomorphism of R-modules. If $f : F \to N$ is any homomorphism such that* $\operatorname{im}(f) \subset \operatorname{im}(q)$ *then f lifts to a homomorphism $\tilde{f} : F \to M$ making the following diagram commute:*

$$\begin{tikzcd} & M \arrow[d, "q"] \\ F \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & N \end{tikzcd}$$

Proof. Let $\mathcal{B}$ be a basis for F. Since F is free it suffices to define $\hat{f}$ on elements of $\mathcal{B}$. For each $b \in B$ we define $\hat{f}(b)$ to be any element of $q^{-1}(f(b))$; such an element exists since $\operatorname{im}(f) \subset \operatorname{im}(q)$. $\square$

A module satisfying the conclusion of Lemma 2.7 is said to be *projective*; thus free modules are projective. An algebra class will have much more to say about projective modules. We only mention the concept here because it is exactly what is needed to prove the next lemma.

Lemma 2.8. *Suppose E is a free resolution of an R-module M and F is a free resolution of an R-module N. For any surjective homomorphism $\varphi : M \to N$ there exists a chain map $\Phi : E \to F$ which extends φ. Moreover, any two extensions of φ are chain homotopic.*

Proof. Let $e_n : E_n \to E_{n-1}$ and $f_n : F_n \to F_{n-1}$ denote the differentials of the chain complexes E and F and let $\epsilon_E : E_o \to H_0(E)$ and $\epsilon_F : F_o \to H_0(F)$ be the quotient homomorphisms. Since $H_0(E) = M$ and $H_0(F) = N$, we view φ as a homomorphism from $H_0(E)$ to $H_0(F)$. We construct Φ by induction, using the projective property of E_n. To keep our fingers busy, here is a diagram that we will be chasing through:

$$
\begin{tikzcd}
\cdots \arrow[r, "e_3"] & E_2 \arrow[r, "e_2"] \arrow[d, "\Phi_2"] & E_1 \arrow[r, "e_1"] \arrow[d, "\Phi_1"] \arrow[dl, "h_1"'] & E_0 \arrow[r, "\epsilon_E"] \arrow[d, "\Phi_0"] \arrow[dl, "h_0"'] & H_0(E) \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r, "f_3"] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "\epsilon_F"] & H_0(F) \arrow[r] & 0
\end{tikzcd}
$$

The composition $\varphi \circ \epsilon_E$ is surjective since both φ and ϵ_E are. We define $\Phi_0 : E_0 \to F_0$ to be a lift of this surjection. Note that, since $\epsilon_F \circ \Phi_0 \circ e_1 = \varphi \circ \epsilon_E \circ e_1 = 0$, we have $\mathrm{im}(\Phi_0 \circ e_1) \subset \ker(\epsilon_F) = \mathrm{im}(f_1)$. For the induction step, given $\mathrm{im}(\Phi_{n-1} \circ e_n) \subset \mathrm{im}(f_n)$, we define Φ_n to be a lift of $\Phi_{n-1} \circ e_n$ and observe that, since $f_n \circ \Phi_n = \Phi_{n-1} \circ e_n$, we then have $\mathrm{im}(\Phi_n \circ e_{n+1}) \subset \ker(f_n) = \mathrm{im}(f_{n+1})$.

Now assume that we have two chain maps Φ and Φ' which both extend φ. We must construct a chain homotopy between them. Again, we do this by induction. We set $\Delta_n = \Phi_n - \Phi'_n$ and, to avoid special cases, we set $h_{-1} = 0$. Since $\Phi_0 \circ \epsilon_F = \Phi'_0 \circ \epsilon_F = \varphi$ we have $\mathrm{im}(\Delta_0) \subset \ker(\epsilon_F) = \mathrm{im}(f_1)$. Thus we may take $h_0 : E_0 \to F_1$ to be a lift of Δ_0. For the induction step we must to define h_n so that $f_{n+1} \circ h_n = \Delta_n - h_{n-1} \circ e_n$, i.e. so that h_n is a lift of $\Delta_n - h_{n-1} \circ e_n$. To see that such a lift exists we verify that

$$
\begin{aligned}
f_n \circ (\Delta_n - h_{n-1} \circ e_n) &= f_n \circ \Delta_n - f_n \circ h_{n-1} \circ e_n \\
&= \Delta_{n-1} \circ e_n - (\Delta_{n-1} - h_{n-2} \circ e_{n-1}) \circ e_n) \\
&= 0.
\end{aligned}
$$

□

Corollary 2.9. *Suppose that E and F are free resolutions of an R-module A. Then the homology of $E \otimes_R A$ is isomorphic to that of $F \otimes_R A$ and the cohomology of* $\mathrm{Hom}_R(E, A)$ *is isomorphic to that of* $\mathrm{Hom}_R(F, A)$.

Proof. Take φ to be the identity homomorphism of A. We may extend id to get chain maps $\Phi : E \to F$ and $\Psi : F \to E$. The compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are chain-homotopic to the identity by Lemma 2.8. Since the functors $\_ \otimes_R A$ and $\mathrm{Hom}_R(\_\,, A)$ are additive, this implies that $\Phi \otimes \mathrm{id}$ and $\Psi \otimes \mathrm{id}$ are inverses, as are Φ^* and Ψ^*. □

The corollary justifies the following definition.

Definition 2.10. Let A be an R-module. If F is any free resolution of an R-module M we define $\mathrm{Tor}_*(M, A)$ to be the homology of the chain complex $F \otimes_R A$ and $\mathrm{Ext}^*(M, A)$ to be the cohomology of the cochain complex $\mathrm{Hom}_R(F, A)$.

Note that $\mathrm{Tor}_0(M, A) = M \otimes_R A$ and $\mathrm{Ext}^0(M, A) = \mathrm{Hom}_R(M, A)$.

Example 2.11. Once again, let's take $R = \mathbb{Z}$, so we are considering abelian groups, and compute a few examples. Since subgroups of free abelian groups are free, we can take our free resolutions to be presentations.

To compute $\mathrm{Tor}_1(\mathbb{Z}_m, \mathbb{Z})$ and $\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z})$ we begin with the presentation $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}_m \to 0$. Applying $\_ \otimes \mathbb{Z}$ gives an isomorphic chain complex, namely

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z}.$$

Thus $\mathrm{Tor}_1(\mathbb{Z}_m, \mathbb{Z}) = 0$, since the kernel of $x \mapsto mx$ is trivial. Applying $\mathrm{Hom}(\_, \mathbb{Z})$ gives the dual cochain complex,

$$\mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to 0 \to \cdots .$$

Since the cokernel of $x \mapsto mx$ is $\mathbb{Z}_m$, we have $\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$,

Computing $\mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}_n)$ and $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}_n)$, or $\mathrm{Tor}_1(F, \mathbb{Z}_n)$ and $\mathrm{Ext}^1(F, \mathbb{Z}_n)$ for a free module F, is even easier since we start with the trivial presentation $0 \to 0 \to F \xrightarrow{\mathrm{id}} F \to 0$, for which the 1-dimensional chain group is 0. Thus the result is 0 for both of these cases as well.

It is a little more interesting to consider $\mathrm{Tor}_1(\mathbb{Z}_m, \mathbb{Z}_n)$ and $\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}_n)$. Here we start with the presentation $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}_m \to 0$. Applying $\_ \otimes \mathbb{Z}_n$ gives the chain complex $\mathbb{Z}_n \xrightarrow{\times m} \mathbb{Z}_n$. Thus $\mathrm{Tor}_1(\mathbb{Z}_m, \mathbb{Z}_n)$ is the kernel of $\mathbb{Z}_n \xrightarrow{\times m} \mathbb{Z}_n$. Writing $pm + qn = r$, where r is the greatest common divisor (m, n), we see that in $\mathbb{Z}_n$ the image of $x \mapsto mx$ is the cyclic group of order n/r generated by r, and the kernel is a cyclic subgroup of order r. Thus $\mathrm{Tor}_1(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_r$. Applying $\mathrm{Hom}(\_, \mathbb{Z}_n)$ gives the cochain complex $\mathbb{Z}_n \xleftarrow{\times m} \mathbb{Z}_n$. Since the cokernel of $x \mapsto mx$ is again a cyclic group of order r we have $\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_r$.

3. Universal Coefficient Theorems

We now specialize to the case $R = \mathbb{Z}$; we will only work with abelian groups in this section.

Start with a chain complex C of free abelian groups with differential ∂_C:

$$\cdots \to C_n \to C_{n-1} \to \cdots C_1 \to C_0 \to 0.$$

We recall the usual definitions:

$$Z_n = \ker(\partial_C|_{C_n} : C_n \to C_{n-1});$$
$$B_n = \mathrm{im}(\partial_C|_{C_n} : C_n \to C_{n-1});$$
$$H_n = Z_n/B_n.$$

The operation of forming the homology groups of a chain complex can be viewed as a functor from chain complexes and chain maps to graded abelian groups and graded homomorphisms. Let's denote this functor by $\mathcal{H}$. We can think of the groups $H_*(C; G)$ and $H^*(C; G)$ as being obtained by applying the functor $\mathcal{H} \circ \mathcal{F}$ to a chain complex C, where $\mathcal{F}$ denotes one of the additive functors $\_ \otimes G$ or $\mathrm{Hom}(\_, G)$. Our goal in the next two sections is to compare $\mathcal{H} \circ \mathcal{F}(C)$ with $\mathcal{F} \circ \mathcal{H}(C)$. It turns out that the homology functor $\mathcal{H}$ does not quite commute with either of the functors $\mathcal{F}$, but it almost does and the comparison leads to descriptions of $H_*(C; G)$ and $H^*(C; G)$ in terms of $H_*(C)$.

3.1. *Two short exact sequences*

By just rewriting our definitions we obtain two families of short exact sequences:

$$0 \to Z_n \to C_n \to B_{n-1} \to 0. \tag{1}$$

and

$$0 \to B_n \to Z_n \to H_n \to 0 \tag{2}$$

The analysis we will carry out here involves viewing each of these short exact sequences in a slightly tricky way. We will view the presentation (2) as a free resolution of the group H_n.

We will view (1) as a short exact sequence of chain complexes

$$0 \to Z \to C \to B \to 0$$

The chain complex Z has Z_n as its n^{th} graded group and has the zero map as its differential. Since ∂_C is zero on each Z_n, the differential of Z is just the restriction of ∂_C. The quotient chain complex B has as its n^{th} graded group B_{n-1} (note the shift by 1!). Since ∂_C is also zero on B_n, the differential on B will be the zero map as well.

This gives the following diagram, which is easily checked to be commutative:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & Z_{n+1} \arrow[r, "0"] \arrow[d] & Z_n \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_C"] \arrow[d, "\partial_C"] & C_n \arrow[r] \arrow[d, "\partial_C"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "0"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

3.2. *A long exact sequence*

Since the chain complexes Z, C and B are free, when we apply one of our additive functors to the short exact sequence

$$0 \to Z \to C \to B \to 0$$

we obtain a new short exact sequence of chain complexes. Here we examine what can be learned from the associated long exact sequence. In the case of $\_\otimes G$ we have

$$H_{n+1}(B;G) \xrightarrow{\partial} H_n(Z;G) \longrightarrow H_n(C;G) \longrightarrow H_n(B;G) \xrightarrow{\partial} H_{n-1}(Z;G)$$

Since the chain complexes B and Z have trivial differential, the same is true of $B\otimes G$ and $Z\otimes G$. For a chain complex with trivial differential, the homology groups are the same as the chain groups. Thus, after accounting for the shift in grading for the chain complex B, we can rewrite the sequence above as

$$B_n\otimes G \xrightarrow{\partial} Z_n\otimes G \longrightarrow H_n(C;G) \longrightarrow B_{n-1}\otimes G \xrightarrow{\partial} Z_{n-1}\otimes G$$

As happens with any long exact sequence, we obtain a sequence of short exact sequences which describe the groups $H_n(C;G)$ as extensions:

$$0 \to \operatorname{coker}(\partial_n) \to H_n(C;G) \to \ker(\partial_{n-1}) \to 0, \tag{3.2.1}$$

where $\partial_k : B_k\otimes G \to Z_k\otimes G$ denotes the connecting homomorphism of the long exact sequence.

If we look closely we can see that this short exact sequence splits, which implies that the group $H_n(C;G)$ is the direct sum of the cokernel of ∂_n and the kernel of ∂_{n-1}. To see this we first observe that the sequence (1) splits, since the abelian groups Z_n, B_n and C_n are all free. This implies that there is a projection $p : C_n \to Z_n$ such that the composition of p with the inclusion $j : Z_n \to C_n$ is the identity on Z_n. After applying $\_\otimes G$ we have $(p\otimes \mathrm{id})\circ(j\otimes\mathrm{id}) = \mathrm{id}$. Restricting $p\otimes\mathrm{id}$ to cycles in $C_n\otimes G$ we get an induced homomorphism $p_* : H_n(C;G) \to \operatorname{coker}(\partial)$ such that $p_*\circ j_* = \mathrm{id}$. This gives a splitting.

It is important to observe that this splitting is not "natural". It can only be constructed at the chain level; it is not necessarily preserved by the homomorphisms induced on homology by chain maps.

If we apply the functor $\mathrm{Hom}(\_,G)$ to the same chain complex C and proceed in a completely analogous manner using the long exact cohomology sequence, we obtain the split short exact sequence

$$0 \to \operatorname{coker}(\delta_{n-1}) \to H^n(C;G) \to \ker(\delta_n) \to 0, \tag{3.2.2}$$

where $\delta_k : \mathrm{Hom}(Z_k,G) \to \mathrm{Hom}(B_k,G)$ denotes the connecting homomorphism of the long exact sequence.

3.3. *A short free resolution*

Next, by using the short exact sequence (2) we will show that $\operatorname{coker}(\partial_n)$ can be identified with $H_n\otimes G$ while $\ker(\partial_{n-1})$ can be identified with $\mathrm{Tor}_1(H_{n-1},G)$. Similarly we will show that $\ker(\delta_n)$ can be identified with $\mathrm{Hom}(H_n,G)$ and $\operatorname{coker}(\delta_{n-1})$ can be identified with $\mathrm{Ext}^1(H_n,G)$.

When we apply our additive functors to the presentation (2), viewed as a free resolution of H_n, we obtain the following two exact sequences:

$$0 \to \mathrm{Hom}(H_n, G) \to \mathrm{Hom}(C_n, G) \xrightarrow{i^*} \mathrm{Hom}(Z_n, G) \to \mathrm{Ext}^1(H_n, G) \to 0 \tag{3.3.1}$$

and

$$0 \to \mathrm{Tor}_1(H_n, G) \to Z_n \otimes G \xrightarrow{i_*} C_n \otimes G \to H_n \otimes G \to 0 \tag{3.3.2}$$

Thus the descriptions of the kernel and cokernel of ∂ and δ will follow from the following lemma.

Proposition 3.4. *Let ∂ and δ denote the connecting homomorphisms from the long exact sequences discussed in section 3.2. Let $i_* : Z \otimes G \to C \otimes G$ and $i^* : \mathrm{Hom}(C, G) \to \mathrm{Hom}(Z, G)$ be induced by the inclusion $i : Z \to C$. Then we have $i_* = \partial$ and $i^* = \delta$.*

Proof. The proof is tautological; we just recall the construction of the connecting homomorphism in the long exact sequence. In the case of $\_ \otimes G$ the procedure is to take a class represented by a generator $b \otimes g \in B_n \otimes G$; choose an element $x \in C_{n+1} \otimes G$ such that $\partial_C \otimes \mathrm{id}(x) = b \otimes g$; and apply $\partial_C \otimes \mathrm{id}$ to x, giving us back $b \otimes g$. We observe that this element is in fact contained in $\mathbb{Z}_n \otimes g$, and define its class to be image of the class of $b \otimes g$. Thus the connecting homomomorphism is nothing other than the homomorphism induced by inclusion.

For $\mathrm{Hom}(\_ , G)$ the construction of the connecting homomorphism is to take a representative cocycle $f \in \mathrm{Hom}(Z_n, G)$; extend it to $\hat{f} \in \mathrm{Hom}(C_n, G)$ using the splitting projection $C_n \to Z_n$; and apply δ to get $\hat{f} \circ \partial_C$. Next we find an element of $\mathrm{Hom}(B_n, G)$ which pulls back to $\hat{f} \circ \partial_C$ under ∂_C. The restriction of f to B_n is such an element. So, again, the connecting homomorphism is induced by inclusion. $\square$

We end the story with the statements of the universal coefficient theorems for homology and cohomology. (Homology and cohomology groups with negative dimensions are taken to be 0 here.)

Theorem 3.5. *Let C be a chain complex and G an abelian group. Then for each integer n there is a split short exact sequence*

$$0 \to \mathrm{Tor}_1(H_{n-1}(C), G) \to H_n(C; G) \to H_n(C) \otimes G \to 0.$$

Proof. According to (3.3.2) we have $\mathrm{Tor}_1(H_{n-1}(C), G) = \ker(i_*)$ and $H_n \otimes G = \mathrm{coker}(i_*)$. But according to Proposition 3.4 we have $i_* = \partial$. Thus the theorem follows from (3.2.1). $\square$

Theorem 3.6. *Let C be a chain complex and G an abelian group. Then for each integer n there is a split short exact sequence*

$$0 \to \mathrm{Ext}_1(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0.$$

Proof. According to (3.3.1) we have $\mathrm{Ext}^1(H_{n-1}(C), G) = \mathrm{coker}(i^*)$ and $\mathrm{Hom}(H_n, G) = \ker(i^*)$. But according to Proposition 3.4 we have $i^* = \delta$. Thus the theorem follows from (3.2.2). $\square$