Assignment #1

1.1.3. Let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$ and $Z \subset \mathbb{R}^L$ be arbitrary subsets, and let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Then the composition $g \circ f$ is smooth. If $f$ and $g$ are diffeomorphisms, so is $g \circ f$.

Solution (by David Chen). Let $f : X \to Y$, $g : Y \to Z$ be smooth maps of $X \subseteq \mathbb{R}^N$, $Y \subseteq \mathbb{R}^M$, $Z \subseteq \mathbb{R}^L$. Take $x_0 \in X$ and $f(x_0) \in Y$. Then there is an open set $V \subseteq \mathbb{R}^M$ on which $g$ extends to a smooth map. Around $x_0$ is an open set $U \subseteq \mathbb{R}^N$ on which $f$ extends to a smooth map, so taking $U \cap f^{-1}(V)$ is an open set on which $g \circ f$ has a smooth extension.

By the chain rule, for $1 \leq k \leq M$, $1 \leq j \leq L$, $1 \leq i \leq N$, we have

$$\frac{\partial g_j}{\partial x_i} = \sum_k \frac{\partial g_j}{\partial f_k} \frac{\partial f_k}{\partial x_i}.$$ 

This shows $g \circ f$ is smooth.

If $f$, $g$ are in fact diffeomorphisms, then $g \circ f$ is bijective and smooth, being a composition of smooth maps. Moreover, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is also a composition of smooth maps, and hence is smooth. Thus $g \circ f$ is also a diffeomorphism.

1.1.12. Stereographic projection is a map $\pi$ from the punctured sphere $S^2 - N$ onto $\mathbb{R}^2$, where $N$ is the north pole $(0, 0, 1)$. For any $p \in S^2 - N$, $\pi(p)$ is defined to be the point at which the line through $N$ and $p$ intersects the $xy$ plane. Prove that $\pi : S^2 - N \to \mathbb{R}^2$ is a diffeomorphism (To do so, write $\pi$ explicitly in coordinates and solve for $\pi^{-1}$). Note that if $p$ is near $N$, then $|\pi(p)|$ is large. Thus $\pi$ allows us to think of $S^2$ as a copy of $\mathbb{R}^2$ compactified by the addition of one point "at infinity". Since we can define stereographic projection by using the south pole instead of the north, $S^2$ may be covered by two local parametrizations.

Solution (by Jessica Dyer). Let $l$ be the line through the point $N$ and a point $(x, y, z)$ on $S^2$. Parametrize the line $l$ by

$$((1 - t) \cdot 0 + tx, (1 - t) \cdot 0 + ty, (1 - t) \cdot 1 + tz) = (tx, ty, 1 - t + tz).$$

To solve for the point where $l$ intersects the $xy$ plane, set the last coordinate equal to zero, so we get $1 - t + tz = 0$, so $t = \frac{1}{1 + tz}$. Thus

$$\pi(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}, 0\right).$$

To get the inverse function, we want to start with the parametrized line through $(x, y, 0)$ and $N$, parametrized by

$$((1 - t) \cdot 0 + tx, (1 - t) \cdot 0 + ty, (1 - t) \cdot 1 + t0) = (tx, ty, 1 - t),$$
and solve for the point on $S^2$, i.e. the point at distance 1 from the origin. Set $(tx)^2 + (ty)^2 + (1 - t)^2 = 1$. Solving gives either $t = 0$, which is the solution at $N$, or $t = \frac{2}{x^2 + y^2 + 1}$. So

$$\pi^{-1}(x, y, 0) = (\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, -1 + x^2 + y^2)$$

Since $\pi$ is a rational function it has continuous partials of all orders except where $1 - z = 0$. The only point on $S^2$ with $z = 1$ is $N$, which is excluded from our domain. So for all points in the domain, $\pi$ has continuous partials. We exhibited $\pi^{-1}$, so $\pi$ is 1-1, and onto by construction. Since $\pi^{-1}$ is also rational, it has continuous partials of all orders by the quotient rule, so $\pi$ is a diffeomorphism.

**1.1.17.** The graph of a map $f : X \to Y$ is the subset $X \times Y$ defined by

$$\text{graph}(f) := \{(x, f(x)) : x \in X\}.$$  

Define $F : X \to \text{graph}(f)$ by $F(x) := (x, f(x))$. Show that if $f$ is smooth, $F$ is a diffeomorphism.

**Solution** (by Landon Kavlie). Using the inverse $F^{-1}(x, y) = x$, we see that $F$ is a bijection. If $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ then $\text{graph}(f) \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Since $F^{-1}$ is the restriction of the linear projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, we conclude that $F^{-1}$ is smooth by problem 1.1.5.

Since $f : X \to Y$ is smooth and the identity function $\text{id}_X : X \to X$ is smooth (it extends to the linear function $\text{id}_{\mathbb{R}^n}$), it is enough to show that for $f : X \to X', g : Y \to Y'$ smooth, the map $f \times g : X \times Y \to X' \times Y'$ given by $(f \times g)(x, y) := (f(x), g(x))$ is smooth.

Since $f : X \to X'$ is smooth, there is a neighborhood $U$ of $x$ and $F : U \to \mathbb{R}^m$ smooth so that $F|_U = f$. Similarly, $y \in Y$ and $g : Y \to Y'$ smooth gives a neighborhood $V$ of $y$ and $G : V \to \mathbb{R}^n$ smooth so that $G|_V = g$. Then, $U \times V$ is an open neighborhood of $(x, y)$. Also, $(F \times G) : U \times V \to \mathbb{R}^{m+n}$ given by $(F \times G)(x, y) := (F(x), G(y))$ is smooth since its first partial derivatives are smooth. That is, for $i = 1, \ldots, n, j = 1, \ldots, m$, $z := (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) = (z_1, \ldots, z_{n+m})$,

$$\frac{\partial(F \times G)}{\partial z_i} = \frac{\partial}{\partial x_i}(F(x), G(x)) = \left(\frac{\partial F}{\partial x_i}(x), 0\right)$$

$$\frac{\partial(F \times G)}{\partial z_{n+j}} = \frac{\partial}{\partial y_j}(F(x), G(x)) = \left(0, \frac{\partial G}{\partial y_j}(y)\right).$$

Finally, since $F \times G|_{U \times V} = f \times g$, we have that $f \times g$ is smooth.

**1.2.1.** For a submanifold $X$ of $Y$, let $i : X \to Y$ be the inclusion. Check that $di_x$ is the inclusion map of $T_x(X)$ into $T_x(Y)$.

**Solution** (by Alex Stathis). Since $X$ sits inside of $Y$ and therefore inside of $\mathbb{R}^N$, for some $N$, we see that $i$ is merely the restriction of the identity map on $\mathbb{R}^N$ (or some small open set around a given point in $X$) to $X$. Since the identity is linear, it follows that $di_x = I$ and that $di_x$ is the restriction of $I$ to the tangent space of $X$. Since we know that $di_x$ maps the
tangent space of $X$ to the tangent space of $T_x(Y)$, we have that $di_x$ is an inclusion of $T_x(X)$ into $T_x(Y)$.

1.2.4. Suppose that $f : X \rightarrow Y$ is a diffeomorphism, and prove that at each $x$ its derivative $df_x$ is an isomorphism of tangent spaces.

**Solution** (by Maxwell Levine). Let $f : X \rightarrow Y$ be a diffeomorphism and let $x \in X$. $f$ has a smooth inverse $f^{-1}$. By the chain rule,

\[ di_x = d(f^{-1} \circ f)_x = df^{-1}_{f(x)} \circ df_x \implies df^{-1}_{f(x)} = (df_x)^{-1} \]

so we get

\[ (df_x)^{-1} \circ (df_x) = df^{-1}_{f(x)} \circ df_x = d(f^{-1} \circ f)_x = di_x \]

di$_x$ is the identity matrix, so $df_x$ is invertible as a linear map. Therefore, it is bijective (by linear algebra) and it is an isomorphism of the linear spaces $T_x(X)$ and $T_{f(x)}(Y)$.

1.2.12.

(a) A curve in a manifold $X$ is a smooth map $t \rightarrow c(t)$ of an interval of $\mathbb{R}$ into $X$. The velocity vector of the curve $c$ at time $t_0$ - denoted simply $dc/dt(t_0)$ - is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $x_0 = c(t_0)$ and $dc_{t_0} : \mathbb{R} \rightarrow T_{x_0}(X)$. In case $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$ in coordinates, check that

\[ \frac{dc}{dt}(t_0) = (c_1'(t_0), \ldots, c_k'(t_0)). \]

(b) Prove that every vector in $T_{x_0}(X)$ is the velocity vector of some curve in $X$, and conversely.

**Solution** (by Ryan Steinbach).

(a)

\[ \frac{dc}{dt}(t_0) = dc_{t_0}(1) \]

\[ = \lim_{t \to 0} \frac{c(t_0 + t) - c(t_0)}{t} \]

\[ = \lim_{t \to 0} \left( \frac{c_1(t_0 + t) - c_1(t_0)}{t}, \ldots, \frac{c_k(t_0 + t) - c_k(t_0)}{t} \right) \]

\[ = (c_1'(t), \ldots, c_k'(t)). \]

(b) ($\Rightarrow$) Let $\vec{v} \in T_{x_0}(X)$ and $\phi : U \rightarrow X$ be a parametrization of an open neighborhood $U$ around $x_0$ with $\phi(0) = x_0$. Then $d\phi_0$ is an isomorphism since $\phi$ is a diffeomorphism. Let $\vec{w} = d\phi_0^{-1}(\vec{v})$. Define $c(t) = \phi(\vec{w}t)$ then $c(t)$ is a curve in $X$ with $x_0 = c(0)$. 

Now, 
\[ \frac{dc}{dt}(0) = dc_0(1) = d\phi_0(\vec{w}) = \vec{v} \]

(⇐) If \( \vec{v} \) is the velocity vector of a curve \( c(t) \) in \( X \) then by definition \( \vec{v} = dc_0(1) \in T_{x_0}(X) \).

1.3.11.

(a) Suppose that \( f : X \to Y \) is a smooth map, and let \( F : X \to X \times Y \) be given by \( F(x) = (x, f(x)) \). Show that \( dF_x(v) = (v, df_x(v)) \).

(b) Prove that the tangent space to \( \text{graph}(f) \) at the point \( (x, f(x)) \) is the graph of \( df_x : T_x(X) \to T_{f(x)}(Y) \).

**Solution** (by Nick Cahill).

(a) By the result of exercise 9, given smooth functions \( f : X \to X' \) and \( g : Y \to Y' \), we have \( d(f \times g)_{(x,y)} = df_x \times dg_y \). For the function \( F : X \to X \times Y \) defined by \( x \mapsto (x, f(x)) \) we therefore have

\[ dF_x = d(x, f(x))_{(x,x)} = (dl_x, df_x) = (I, df_x), \]

where \( I \) denotes the identity map. Thus \( dF_x(v) = (v, df_x(v)) \), as needed.

(b) The map \( F \) defined above gives a parametrization of \( \text{graph}(f) \). Thus, by definition, the tangent space to \( \text{graph}(f) \) at a point \( (x, f(x)) \) is

\[ dF_x(T_x(X)) = T_x(X) \times df_x(T_x(X)). \]

The graph of \( df_x : T_x(X) \to T_{f(x)}(Y) \) consists of all ordered pairs \( (v, df_x(v)) \) for \( v \in T_x(X) \). Thus the graph of \( df_x \) is equal to the subspace \( T_x(X) \times df_x(T_x(X)) \) of \( T_x(X) \times T_{f(x)}(Y) \); we have already observed that this is equal to the tangent space to \( \text{graph}(f) \) at \( (x, f(x)) \).