1.6.3. Show that every connected manifold $X$ is arcwise connected: given any two points $x_0, x_1 \in X$, there exists a smooth curve $f : I \to X$ with $f(0) = x_0, f(1) = x_1$.

Solution (by Sam Ziegler). Let $x_0, x_1, x_2 \in X$, and let $f : I \to X$ be smooth such that $f(0) = x_0, f(1) = x_1$. Also let $g : I \to X$ with $g(0) = x_1, g(1) = x_2$. We can view $f, g$ as homotopies $\{x\} \times I \to X$, where $f_t(x) = f(t), g_t(x) = g(t)$. According to exercise 2 of this section, homotopy is an equivalence relation, so there is a homotopy $h : \{x\} \times I \to X$ with $h_0(x) = x_0$ and $h_1(x) = x_2$. So $h(t) := h_t(x)$ is a smooth map $I \to X$ witnessing that $x_0$ and $x_2$ are connected by a path. Thus, the relation of being connected by a path is transitive. Since reflexivity and symmetry are clear, the relation $x \sim y \leftrightarrow x$ is connected by a smooth path to $y$ is an equivalence relation. Moreover, equivalence classes of the relation are open: given $x_0 \in X$, there is an open set $x \in B \subseteq X$ such that $B$ is diffeomorphic to $\mathbb{R}^k$, which is arcwise connected, hence $B$ is arcwise connected. Finally, if $[x_0]$, the class of $x_0$, is not equal to all of $X$, then its complement is the (nonempty) union of all other equivalence classes, which is also open. Then $X$ is disconnected.