QUOTIENT SPACES – MATH 446

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Throughout these notes, a *map* from a topological space X to a topological space Y will always mean a continuous function from X to Y.

1. Partitions

Definition 1.1. A *partition* of a set X is a collection $\mathcal{P}$ of non-empty subsets of X such that

- For all $U, V \in \mathcal{P}$, either $U = V$ or $U \cap V = \emptyset$; and
- $\bigcup_{U \in \mathcal{P}} U = X$.

Ir $\mathcal{P}$ is a partition of X then the *canonical map* $q : X \to \mathcal{P}$ is the unique function such that $x \in q(x)$ for each $x \in X$.

There are two (related) situations which naturally give rise to partitions:

- If $\sim$ is an equivalence relation on X, then the collection of equivalence classes under $\sim$ forms a partition of X and the canonical map sends each element of X to the equivalence class that contains it.
- If $f : X \to Y$ is a surjective function from X to a set Y, then the collection

$$\mathcal{P} = \{f^{-1}(y) \mid y \in Y\}$$

  of all point-inverses forms a partition of X, and the canonical map sends $x \in X$ to the subset $f^{-1}(f(x))$.

Exercise 1.1. Suppose $f : X \to Y$ is a surjection. For points $x_1, x_2 \in X$, define $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Show that $\sim$ is an equivalence relation and that there is a natural one-to-one correspondence between equivalence classes of $\sim$ and points of Y.

Exercise 1.2. Suppose $\sim$ is an equivalence relation on a set X. Let Y denote the set of equivalence classes for the relation $\sim$ and define a map $q : X \to Y$ by $q(x) = [x]$, where $[x]$ denotes the equivalence class containing x. Show that q is a surjection and that the point-inverses of q are exactly the equivalence classes.

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Definition 1.2. Suppose that X is a topological space, and we are given a partition $\mathcal{P}$ of X. For any subset $\mathcal{S}$ of $\mathcal{P}$ we denote by $\cup\mathcal{S}$ the union of all of the elements of $\mathcal{S}$ (each of which is a subset of X). That is,

$$\cup\mathcal{S} = \bigcup_{S\in\mathcal{S}} S.$$

2. Quotient topologies and quotient maps

Definition 2.1. Let X be a topological space and $\mathcal{P}$ a partition of X. The *quotient topology* on $\mathcal{P}$ is the collection

$$\mathcal{T} = \{\mathcal{O} \subset \mathcal{P} \mid \cup\mathcal{O} \text{ is open in } X\}.$$

Thus the open sets in the quotient topology are collections of subsets whose union is open in X. We can think of the partition elements as "fat points", and the open sets as collections of "fat points" whose union is open as a subset of X.

Example 2.2. For a non-negative integer n, consider the subspace $X = \mathbb{R}^{n+1} - \{0\}$ of $\mathbb{R}^{n+1}$ with its standard topology. Define an equivalence relation on X by $\vec{u} \sim \vec{v}$ if there exists $t \in \mathbb{R}$ such that $\vec{u} = t\vec{v}$. Thus an equivalences class is the intersection of X with a line through the origin in $\mathbb{R}^{n+1}$. The quotient space determined by $\sim$ is called the *real projective space* of dimension n, and is denoted $\mathbb{P}^n$. Since the "fat points" of $\mathbb{P}^n$ are (deleted) lines through the origin, we may think of $\mathbb{P}^n$ as topologizing the set of lines through the origin.

Definition 2.3. Let $q : X \to Y$ be a surjective map between topological spaces. Let $\mathcal{P}$ be the partition of X consisting of point-inverses for q. Give $\mathcal{P}$ the quotient topology. Let $q^* : Y \to \mathcal{P}$ be the natural bijection defined by $q^*(y) = q^{-1}(y)$. If the bijection q^* is a homeomorphism then q is said to be a *quotient map*.

Exercise 2.1. Suppose that $f : X \to Y$ is a surjective map between topological spaces. Show that f is a quotient map if and only if it has the following equivalent properties:

- A subset O of Y is open if and only if $f^{-1}(O)$ is open.
- A subset C of Y is closed if and only if $f^{-1}(C)$ is closed.

Exercise 2.2. Suppose that $q : X \to Y$ is a quotient map. Prove that the topology on Y is the largest topology which makes q continuous.

Proposition 2.4. *Suppose that $f : X \to Y$ is a surjective map between topological spaces. If f is an open map or if f is a closed map, then f is a quotient map.*

Proof. First, suppose that f is an open map. Let O be a subset of Y such that $f^{-1}(O)$ is open. Since f is surjective, we have $f(f^{-1}(O)) = O$. Thus O is open if $f^{-1}(O)$ is open, which shows that f is a quotient map.

Next, suppose that f is a closed map. Let O be a subset of Y such that $f^{-1}(O)$ is open. Since $f^{-1}(X - O) = X - f^{-1}(O)$, $f^{-1}(X - O)$ is closed. But, since f is surjective, we have $f(f^{-1}(X - 0)) = X - O$. Thus $X - O$ is closed, and hence O is open. Again, this shows that f is a quotient map. □

The next two examples show that a quotient map need not be open or closed.

Example 2.5. Consider the partition $\mathcal{P}$ of $\mathbb{R}$ given as follows:

$$\mathcal{P} = \{(0, 1)\} \cup \{\{x\} \mid x \leq 0 \text{ or } x \leq 1\},$$

and give $\mathcal{P}$ the quotient topology. The subset $(0, 1)$ of $\mathbb{R}$ becomes a point of $\mathcal{P}$, and in the quotient topology the singleton set $\{(0, 1)\}$ is open. The point $\{0\} \in \mathcal{P}$ has the property that every neighborhood of $\{0\}$ contains the point $(0, 1) \in \mathcal{P}$. In particular, this shows that the singleton set $\{(0, 1)\}$ is not closed. But the quotient map $q : \mathbb{R} \to \mathcal{P}$ sends the closed singleton set $\{\frac{1}{2}\}$ to the singleton set $\{(0, 1)\}$, which is not closed. Thus q is not a closed map.

Exercise 2.3. Show that a quotient space of a Hausdorff space need not be Hausdorff.

Example 2.6. Consider the partition $\mathcal{P}$ of $\mathbb{R}$ given as follows:

$$\mathcal{P} = \{[0, 1]\} \cup \{\{x\} \mid x < 0 \text{ or } x > 1\},$$

and give $\mathcal{P}$ the quotient topology. The subset $[0, 1]$ of $\mathbb{R}$ is a point of $\mathcal{P}$, and in the quotient topology the singleton set $\{[0, 1]\}$ is open. The point $\{0\} \in \mathcal{P}$ has the property that every neighborhood of $\{0\}$ contains the point $(0, 1) \in \mathcal{P}$. In particular, this shows that the singleton set $\{(0, 1)\}$ is not open. But the quotient map $q : \mathbb{R} \to \mathcal{P}$ sends the open set $(0, 1)$ to the singleton set $\{(0, 1)\}$, which is not open. Thus q is not an open map.

3. Reducing a quotient

Suppose $q : X \to Y$ is a quotient map. It is often useful to have a simpler description of Y, where Y is described as a quotient of a subspace of X. More generally, given any subspace $A \subset X$, we may view the restriction $f|_A$ of f to A as a surjective function from A to $f(A)$. It is natural to ask whether $f|_A : A \to f(A)$ is a quotient map.

The following proposition gives an answer in some useful special cases.

Proposition 3.1. *Suppose that $f : X \to Y$ is a quotient map and $A \subset X$ is a subspace. The surjective function $f|_A : A \to f(A)$ is a quotient map from the subspace A of X to the subspace $f(A)$ of Y in either of the following two cases:*

- *A is a closed set in X and f is a closed map.*
- *A is an open set in X and f is an open map.*

Proof. First consider the case where A and f are closed. Since A is closed, a subset of A is closed in the subspace topology of A if and only if it is closed in the topology of X. Since f is closed, the set $f(A)$ is closed in Y, and a subset of $f(A)$ is closed in the subspace topology of $f(A)$ if and only if it is closed in Y. If $K \subset A$ is closed in the subspace A then it is closed in X; thus $f(K) \subset f(A)$ is closed in Y and hence in the subspace $f(A)$. This shows that $f|_A$ is a closed map, and hence a quotient map by Proposition 2.4.

The proof of the second case is the same with the word "closed" replaced by "open". ☐

Exercise 3.1. The n-sphere is the subspace $S^n = \{\vec{v} \in \mathbb{R}^{n+1} \mid ||v|| = 1\}$ of $\mathbb{R}^{n+1}$. Define an equivalence relation on S^n by $u \sim v$ if and only if $u = \pm v$, so each equivalence class consists of two antipodal points on the sphere. Prove that the quotient space determined by $\sim$ is homeomorphic to $\mathbb{P}^n$.

Exercise 3.2. Define an equivalence relation on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 - x_2$ and $y_1 - y_2$ are integers. Let T denote the quotient space determined by $\sim$. Describe T as a quotient of the unit square $[0, 1] \times [0, 1]$.

Exercise 3.3. Consider the partition $\mathcal{P}$ of the unit disk $D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ given by

$$\mathcal{P} = \{\{(x, y) \mid x^2 + y^2 \leq 1/4\}\} \cup \{\{(x, y)\} \mid 1/4 < x^2 + y^2 \leq 1\}.$$

(So all but one of the elements of $\mathcal{P}$ are singleton sets, and the exceptional element is the disk of radius 1/2.) Prove that the quotient space determined by $\mathcal{P}$ is homeomorphic to D^2.

Exercise 3.4. Consider the partition $\mathcal{P}$ of D^2 in which the unit circle is one of the sets in the partition and the others are all singleton sets. Prove that the quotient space determined by $\mathcal{P}$ is homeomorphic to S^2, the unit sphere in $\mathbb{R}^3$.