Throughout these notes, a map from a topological space $X$ to a topological space $Y$ will always mean a continuous function from $X$ to $Y$.

1. Partitions

**Definition 1.1.** A partition of a set $X$ is a collection $\mathcal{P}$ of non-empty subsets of $X$ such that

- For all $U, V \in \mathcal{P}$, either $U = V$ or $U \cap V = \emptyset$; and
- $\bigcup_{U \in \mathcal{P}} U = X$.

If $\mathcal{P}$ is a partition of $X$ then the canonical map $q : X \to \mathcal{P}$ is the unique function such that $x \in q(x)$ for each $x \in X$.

There are two (related) situations which naturally give rise to partitions:

- If $\sim$ is an equivalence relation on $X$, then the collection of equivalence classes under $\sim$ forms a partition of $X$ and the canonical map sends each element of $X$ to the equivalence class that contains it.
- If $f : X \to Y$ is a surjective function from $X$ to a set $Y$, then the collection
  $$\mathcal{P} = \{f^{-1}(y) \mid y \in Y\}$$
  of all point-inverses forms a partition of $X$, and the canonical map sends $x \in X$ to the subset $f^{-1}(f(x))$.

**Exercise 1.1.** Suppose $f : X \to Y$ is a surjection. For points $x_1, x_2 \in X$, define $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Show that $\sim$ is an equivalence relation and that there is a natural one-to-one correspondence between equivalence classes of $\sim$ and points of $Y$.

**Exercise 1.2.** Suppose $\sim$ is an equivalence relation on a set $X$. Let $Y$ denote the set of equivalence classes for the relation $\sim$ and define a map $q : X \to Y$ by $q(x) = [x]$, where $[x]$ denotes the equivalence class containing $x$. Show that $q$ is a surjection and that the point-inverses of $q$ are exactly the equivalence classes.

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Definition 1.2. Suppose that \( X \) is a topological space, and we are given a partition \( \mathcal{P} \) of \( X \). For any subset \( S \) of \( \mathcal{P} \) we denote by \( \cup S \) the union of all of the elements of \( S \) (each of which is a subset of \( X \)). That is,
\[
\cup S = \bigcup_{S \in \mathcal{S}} S.
\]

2. Quotient topologies and quotient maps

Definition 2.1. Let \( X \) be a topological space and \( \mathcal{P} \) a partition of \( X \). The quotient topology on \( \mathcal{P} \) is the collection
\[
\mathcal{T} = \{ \mathcal{O} \subset \mathcal{P} | \cup \mathcal{O} \text{ is open in } X \}.
\]
Thus the open sets in the quotient topology are collections of subsets whose union is open in \( X \). We can think of the partition elements as “fat points”, and the open sets as collections of “fat points” whose union is open as a subset of \( X \).

Example 2.2. For a non-negative integer \( n \), consider the subspace \( X = \mathbb{R}^{n+1} - \{0\} \) of \( \mathbb{R}^{n+1} \) with its standard topology. Define an equivalence relation on \( X \) by \( \vec{u} \sim \vec{v} \) if there exists \( t \in \mathbb{R} \) such that \( \vec{u} = t\vec{v} \). Thus an equivalences class is the intersection of \( X \) with a line through the origin in \( \mathbb{R}^{n+1} \). The quotient space determined by \( \sim \) is called the real projective space of dimension \( n \), and is denoted \( \mathbb{P}^n \). Since the “fat points” of \( \mathbb{P}^n \) are (deleted) lines through the origin, we may think of \( \mathbb{P}^n \) as topologizing the set of lines through the origin.

Definition 2.3. Let \( q : X \to Y \) be a surjective map between topological spaces. Let \( \mathcal{P} \) be the partition of \( X \) consisting of point-inverses for \( q \). Give \( \mathcal{P} \) the quotient topology. Let \( q^* : Y \to \mathcal{P} \) be the natural bijection defined by \( q^*(y) = q^{-1}(y) \). If the bijection \( q^* \) is a homeomorphism then \( q \) is said to be a quotient map.

Exercise 2.1. Suppose that \( f : X \to Y \) is a surjective map between topological spaces. Show that \( f \) is a quotient map if and only if it has the following equivalent properties:

- A subset \( O \) of \( Y \) is open if and only if \( f^{-1}(O) \) is open.
- A subset \( C \) of \( Y \) is closed if and only if \( f^{-1}(C) \) is closed.

Exercise 2.2. Suppose that \( q : X \to Y \) is a quotient map. Prove that the topology on \( Y \) is the largest topology which makes \( q \) continuous.

Proposition 2.4. Suppose that \( f : X \to Y \) is a surjective map between topological spaces. If \( f \) is an open map or if \( f \) is a closed map, then \( f \) is a quotient map.
Proof. First, suppose that \( f \) is an open map. Let \( O \) be a subset of \( Y \) such that \( f^{-1}(O) \) is open. Since \( f \) is surjective, we have \( f(f^{-1}(O)) = O \). Thus \( O \) is open if \( f^{-1}(O) \) is open, which shows that \( f \) is a quotient map.

Next, suppose that \( f \) is a closed map. Let \( O \) be a subset of \( Y \) such that \( f^{-1}(O) \) is open. Since \( f^{-1}(X - O) = X - f^{-1}(O) \), \( f^{-1}(X - O) \) is closed. But, since \( f \) is surjective, we have \( f(f^{-1}(X - 0)) = X - O \). Thus \( X - O \) is closed, and hence \( O \) is open. Again, this shows that \( f \) is a quotient map. \( \square \)

The next two examples show that a quotient map need not be open or closed.

**Example 2.5.** Consider the partition \( \mathcal{P} \) of \( \mathbb{R} \) given as follows:

\[
\mathcal{P} = \{(0, 1)\} \cup \{\{x\} \mid x \leq 0 \text{ or } x \leq 1\},
\]

and give \( \mathcal{P} \) the quotient topology. The subset \((0, 1)\) of \( \mathbb{R} \) becomes a point of \( \mathcal{P} \), and in the quotient topology the singleton set \(\{(0, 1)\}\) is open. The point \(\{0\} \in \mathcal{P}\) has the property that every neighborhood of \(\{0\}\) contains the point \((0, 1) \in \mathcal{P}\). In particular, this shows that the singleton set \(\{(0, 1)\}\) is not closed. But the quotient map \(q : \mathbb{R} \to \mathcal{P}\) sends the closed singleton set \(\{\frac{1}{2}\}\) to the singleton set \(\{(0, 1)\}\), which is not closed. Thus \(q\) is not a closed map.

**Exercise 2.3.** Show that a quotient space of a Hausdorff space need not be Hausdorff.

**Example 2.6.** Consider the partition \( \mathcal{P} \) of \( \mathbb{R} \) given as follows:

\[
\mathcal{P} = \{[0, 1]\} \cup \{\{x\} \mid x < 0 \text{ or } x > 1\},
\]

and give \( \mathcal{P} \) the quotient topology. The subset \([0, 1]\) of \( \mathbb{R} \) is a point of \( \mathcal{P} \), and in the quotient topology the singleton set \(\{[0, 1]\}\) is open. The point \(\{0\} \in \mathcal{P}\) has the property that every neighborhood of \(\{0\}\) contains the point \((0, 1) \in \mathcal{P}\). In particular, this shows that the singleton set \(\{(0, 1)\}\) is not open. But the quotient map \(q : \mathbb{R} \to \mathcal{P}\) sends the open set \((0, 1)\) to the singleton set \(\{(0, 1)\}\), which is not open. Thus \(q\) is not an open map.

### 3. Reducing a quotient

Suppose \( q : X \to Y \) is a quotient map. It is often useful to have a simpler description of \( Y \), where \( Y \) is described as a quotient of a subspace of \( X \). More generally, given any subspace \( A \subset X \), we may view the restriction \( f|_A \) of \( f \) to \( A \) as a surjective function from \( A \) to \( f(A) \). It is natural to ask whether \( f|_A : A \to f(A) \) is a quotient map.

The following proposition gives an answer in some useful special cases.
Proposition 3.1. Suppose that \( f : X \to Y \) is a quotient map and \( A \subset X \) is a subspace. The surjective function \( f|_A : A \to f(A) \) is a quotient map from the subspace \( A \) of \( X \) to the subspace \( f(A) \) of \( Y \) in either of the following two cases:

- \( A \) is a closed set in \( X \) and \( f \) is a closed map.
- \( A \) is an open set in \( X \) and \( f \) is an open map.

Proof. First consider the case where \( A \) and \( f \) are closed. Since \( A \) is closed, a subset of \( A \) is closed in the subspace topology of \( A \) if and only if it is closed in the topology of \( X \). Since \( f \) is closed, the set \( f(A) \) is closed in \( Y \), and a subset of \( f(A) \) is closed in the subspace topology of \( f(A) \) if and only if it is closed in \( Y \). If \( K \subset A \) is closed in the subspace \( A \) then it is closed in \( X \); thus \( f(K) \subset f(A) \) is closed in \( Y \) and hence in the subspace \( f(A) \). This shows that \( f|_A \) is a closed map, and hence a quotient map by Proposition 2.4.

The proof of the second case is the same with the word “closed” replaced by “open”.

Exercise 3.1. The \( n \)-sphere is the subspace \( S^n = \{ \vec{v} \in \mathbb{R}^{n+1} \mid ||\vec{v}|| = 1 \} \) of \( \mathbb{R}^{n+1} \). Define an equivalence relation on \( S^n \) by \( u \sim v \) if and only if \( u = \pm v \), so each equivalence class consists of two antipodal points on the sphere. Prove that the quotient space determined by \( \sim \) is homeomorphic to \( \mathbb{P}^n \).

Exercise 3.2. Define an equivalence relation on \( \mathbb{R}^2 \) by \((x_1, y_1) \sim (x_2, y_2)\) if and only if \( x_1 - x_2 \) and \( y_1 - y_2 \) are integers. Let \( T \) denote the quotient space determined by \( \sim \). Describe \( T \) as a quotient of the unit square \([0, 1] \times [0, 1]\).

Exercise 3.3. Consider the partition \( \mathcal{P} \) of the unit disk \( D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2 \) given by

\[
\mathcal{P} = \{ \{(x, y) \mid x^2 + y^2 \leq 1/4\} \} \cup \{ \{(x, y)\} \mid 1/4 < x^2 + y^2 \leq 1\}.
\]

(So all but one of the elements of \( \mathcal{P} \) are singleton sets, and the exceptional element is the disk of radius \( 1/2 \).) Prove that the quotient space determined by \( \mathcal{P} \) is homeomorphic to \( D^2 \).

Exercise 3.4. Consider the partition \( \mathcal{P} \) of \( D^2 \) in which the unit circle is one of the sets in the partition and the others are all singleton sets. Prove that the quotient space determined by \( \mathcal{P} \) is homeomorphic to \( S^2 \), the unit sphere in \( \mathbb{R}^3 \).