

A WEEK OF RIEMANNIAN MANIFOLDS

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1. Acknowledgement

These notes borrow extensively from the following three books: *Morse Theory*, by John Milnor; *Notes on Differential Geometry*, by Noel J. Hicks; and *Elementary Differential Geometry* by Barrett O'Neill.

2. Introduction

At the very least, a geometric theory of manifolds would include a notion of distance, which could be expected to take the form of a metric that generates the topology of the manifold's underlying topological space. In an interesting geometric theory this metric would be a *path metric*, meaning that there is a notion of the length of a path, and that the distance between two points is the infimum of the lengths of the paths joining them. In fact one would expect to have a notion of "straight lines" or *geodesics*, such that the distance between two points is realized by a path contained in a geodesic.

As we have already seen, a natural way to define a metric in the context of differentiable manifolds is to endow the manifold with the extra structure of a Riemannian metric.

Definition 2.1. A *Riemannian metric* on a smooth n -manifold M is an assignment to each point p in M of a positive definite bilinear pairing $\langle \cdot, \cdot \rangle_p$ on $T_p M$ which is smooth in the sense that if X and Y are smooth vector fields on M then the function $p \mapsto \langle X, Y \rangle_p$ is smooth. For any tangent vector $v \in T_p M$ we will set $\|v\| = \sqrt{\langle v, v \rangle_p}$. For smooth vector fields X and Y we will denote by $\langle X, Y \rangle$ and $\|X\|$ the functions $p \mapsto \langle X, Y \rangle_p$ and $p \mapsto \|X(p)\|$ respectively. A *Riemannian manifold* is a smooth manifold endowed with a Riemannian metric.

A Riemannian metric immediately provides a definition of the arclength of a smooth path $\sigma : [a, b] \rightarrow M$:

$$\text{Length}(\sigma) = \int_a^b \|\sigma'(t)\| dt.$$

It is also immediate that a smooth path with non-vanishing velocity can be reparametrized as a *unit-speed* path, i.e. one for which the velocity vector has length 1 with respect to the Riemannian metric.

There is a straightforward way to describe a Riemannian metric in local coordinates. If (W, ϕ) is a chart with coordinate functions x_1, \dots, x_n then we express $\langle \cdot, \cdot \rangle_p$ in terms of the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ by specifying a positive definite symmetric matrix. Specifically, the metric will be described by a matrix $g = [g_{ij}]$ where each g_{ij} is a smooth function on U and where $g(p) = [g_{ij}(p)]$ is a positive definite symmetric matrix for each $p \in U$.

However we will use a different approach, known as the Cartan viewpoint. Instead of working with the basis determined by the coordinate functions of a chart (W, ϕ) , we will work with a family of smooth vector fields.

Definition 2.2. A *frame field* on an open set W in a smooth n -manifold M is an n -tuple of vector fields (E_1, \dots, E_n) such that $E_1(p), \dots, E_n(p)$ are defined and form a basis for T_pM for each $p \in W$.

Now suppose M is a Riemannian manifold. An *orthonormal frame field* on an open set in M is a frame field (U_1, \dots, U_n) such that the vectors $U_1(p), \dots, U_n(p)$ form an orthonormal basis of T_pM .

If X and Y are arbitrary vector fields given in terms of an orthonormal frame field (U_1, \dots, U_n) by $X = \sum_{i=1}^n a_i U_i$ and $Y = \sum_{i=1}^n b_i U_i$ then we have

$$\langle X, Y \rangle = \sum_{i=1}^n a_i b_i.$$

This is perhaps a good opportunity to adopt the Einstein summation convention: when working with an n -manifold, unspecified summation indices will run from 1 to n , and every sum with unspecified indices will run over all of the indices which appear twice in the summand. For example, we will write

$$\langle X, Y \rangle = \sum a_i b_i.$$

Exercise 2.1. Let M be a Riemannian manifold. Use the Gram-Schmidt process to show that there exists a frame field on any chart neighborhood.

Exercise 2.2. Explain why it would be silly to assume that a frame field is defined on all of M .

Now we come back to the notion of a geodesic, which involves much more subtle ideas than that of the length of a path. Because of our experience with the geometry of \mathbb{R}^n and with surfaces in \mathbb{R}^3 we expect that geodesics will have a dual nature. On the one hand they have a length minimizing property, namely that the distance between two points is realized by a path which lies on a geodesic. On the other hand, we expect geodesics to be paths which are “straight”. For a unit-speed path σ in R^n we can define “straightness” to mean that the acceleration σ'' is 0. A similar idea works for a surface in $\Sigma \subset \mathbb{R}^3$. Given a path σ which lies on Σ for all time, we define its acceleration *relative to* Σ to be the orthogonal projection of

$\sigma''(t)$ into the tangent space $T_{\sigma(t)}\Sigma$, viewed as a subspace of \mathbb{R}^n . If σ is a unit-speed path on Σ then it is a geodesic provided that it has acceleration 0 relative to Σ , i.e. σ is a geodesic if $\|\sigma'(t)\| = 1$ and $\sigma''(t)$ is perpendicular to $T_{\sigma(t)}\Sigma$ for all t .

Subtle problems arise when we try to extend these ideas to an abstract manifold. Computing the acceleration of a path $\sigma : [a, b] \rightarrow M$ involves computing the rate of change of its velocity. This requires that we be able to compute the difference of two velocity vectors $\sigma'(t)$ and $\sigma'(t+h)$. But these vectors lie in two different vector spaces, $T_{\sigma(t)}M$ and $T_{\sigma(t+h)}M$. A priori we have no canonical way of subtracting two such vectors. It can be done for a path in \mathbb{R}^n only because the identification of the tangent bundle of \mathbb{R}^n with $\mathbb{R}^n \times \mathbb{R}^n$ provides a standard way to project all of the tangent spaces onto a single copy of \mathbb{R}^n . There is no standard way of doing this for a general smooth manifold. While each chart provides its own trivialization of an open subset of the tangent bundle, it is not clear which trivialization should be viewed as the “standard” one. Moreover, if such a standard trivialization were to be useful it would have to depend somehow on the Riemannian metric.

3. Connections

The first step towards defining acceleration in a general manifold is to reformulate the definition for \mathbb{R}^n in terms of covariant derivatives. Suppose that X and Y are smooth vector fields defined on an open set in \mathbb{R}^n . The *standard covariant derivative on \mathbb{R}^n* is an operation on a pair of vector fields X and Y which produces a new vector field which we will denote $D_X(Y)$. If we write $X = \sum a_i \frac{\partial}{\partial x_i}$ and $Y = \sum b_j \frac{\partial}{\partial x_j}$, then $D_X(Y)$ is defined by

$$D_X(Y) = \sum X(b_j) \frac{\partial}{\partial x_j} = \sum a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}.$$

It is easily checked that the standard covariant derivative satisfies the following properties, where X, Y and Z are smooth vector fields and f and g are smooth functions.

- $D_{fX+gY}(Z) = fD_X(Z) + gD_Y(Z)$;
- $D_Z(X+Y) = D_Z(X) + D_Z(Y)$; and
- $D_X(fY) = X(f)Y + fD_X(Y) = df(X)Y + fD_X(Y)$

(In particular, D_X is *not* an endomorphism of the $C^\infty(M)$ -module $\mathcal{X}(M)$, i.e. it is not a “tensor”.)

Now if σ is a smooth path in M then, for sufficiently small intervals $[a, b]$ we may view σ' and σ'' as restrictions to $\sigma([a, b])$ of vector fields defined on an open neighborhood of $\sigma([a, b])$ in M , and we then have $\sigma''(t) = D_{\sigma'(t)}(\sigma'(t))$.

To generalize this, we abstract the notion of a covariant derivative.

Definition 3.1. A *covariant derivative* on a smooth manifold M is an operation on $\mathcal{X}(M)$ that combines a pair (X, Y) of vector fields to produce a vector field $\nabla_X(Y)$, such that the following properties are satisfied for $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$:

- $\nabla_{fX+gY}(Z) = f\nabla_X(Z) + g\nabla_Y(Z)$;
- $\nabla_Z(X + Y) = \nabla_Z(X) + \nabla_Z(Y)$; and
- $\nabla_X(fY) = X(f)Y + f\nabla_X(Y) = df(X)Y + f\nabla_X(Y)$

where $X(f) = df(X)$ denotes the directional derivative of the function f along the vector field X .

While this would allow us to make the definition that, for a smooth path σ in M , $\sigma'' = \nabla_{\sigma'(t)}(\sigma'(t))$ we are still a long way from being able to do geometry in M . There are two serious deficiencies in this definition of acceleration. First, it applies to any smooth manifold, and makes no reference to a Riemannian metric. So it cannot possibly carry any geometric information. Second, even after requiring our covariant derivative to preserve the Riemannian structure, there will be not be a unique covariant derivative associated to each metric. Extra conditions are required to obtain a covariant derivative which is uniquely determined by the metric.

The following definition links a covariant derivative to the Riemannian structure on M .

Definition 3.2. Let M be a Riemannian manifold. A covariant derivative on M is *compatible* with the metric if, for any smooth vector fields X, Y and Z defined on an open set in M , we have

$$Z(\langle X, Y \rangle) = \langle \nabla_Z(X), Y \rangle + \langle X, \nabla_Z(Y) \rangle$$

An immediate consequence of this definition is that if X is a unit vector field, and if the covariant derivative ∇ is compatible, then $\langle \nabla_Z(X), X \rangle = 0$. In particular, the acceleration of a unit speed path is perpendicular to the velocity. More generally, a vector field X is said to be *parallel* along a unit speed curve $\sigma(t)$ if $\nabla_{\sigma'}(X) = 0$. The compatibility condition guarantees that if two vector fields are parallel along a curve then the angle between them is constant along the curve.

The uniqueness issue will be dealt with in the next section. For the rest of this section we just focus on how to describe a covariant derivative.

Even though a covariant derivative is not $C^\infty(M)$ -linear, it is nonetheless determined by its action on elements of a frame field. If we are given a frame field (E_1, \dots, E_n) and two arbitrary vector fields $X = \sum a_i E_i$ and $Y = \sum b_i E_i$ then, according to the properties in the

definition, we have

$$\begin{aligned}
\nabla_X(Y) &= \sum a_i \nabla_{E_i}(Y) \\
&= \sum a_i \nabla_{E_i}(b_j E_j) \\
&= \sum a_i E_i(b_j) E_j + \sum a_i b_j \nabla_{E_i}(E_j) \\
&= \sum a_i db_j(E_i) E_j + \sum a_i b_j \nabla_{E_i}(E_j)
\end{aligned}$$

Thus, to describe a covariant derivative in terms of the frame field (E_1, \dots, E_n) , it suffices to give the n^2 vector fields $\nabla_{E_i}(E_j)$. In the classical description in terms of coordinate functions on a chart neighborhood, one takes $E_i = \frac{\partial}{\partial x_i}$ and defines the Christoffel symbols Γ_{ij}^k by the formula

$$\nabla_{E_i}(E_j) = \sum \Gamma_{ij}^k E_k.$$

These n^3 smooth functions determine the covariant derivative, and the Christoffel Identities characterize the collections of n^3 functions which determine a compatible covariant derivative. Since we will not be working with Christoffel symbols, we refer the reader to Milnor's Morse theory book for an efficient treatment.

Using differential forms makes it a bit less daunting to describe the data that determine a covariant derivative. Starting with our frame field (E_1, \dots, E_n) , we may consider the dual 1-forms $\theta_1, \dots, \theta_n$ which are defined by the condition $\theta_i(E_j) = \delta_{ij}$. We then define n^2 1-forms by the formula

$$\omega_{ij}(X) = \theta_j(\nabla_X(E_i)), \quad i, j = 1, \dots, n.$$

These are 1-forms because $\nabla_X(Y)$ is linear in the variable X . Moreover, for each $i = 1, \dots, n$ we have the identity

$$\nabla_X(E_i) = \sum \omega_{ij}(X) E_j.$$

Now if we are given vector fields X and $Y = \sum b_i E_i$ then we have

$$\begin{aligned}
(3.2.1) \quad \nabla_X(Y) &= \sum \nabla_X(b_i E_i) \\
&= \sum db_i(X) E_i + b_i \nabla_X(E_i) \\
&= \sum db_i(X) E_i + \sum b_i \omega_{ij}(X) E_j
\end{aligned}$$

In particular, the covariant derivative is determined by the frame field (E_1, \dots, E_n) and the 1-forms ω_{ij} . We will refer to these forms as *connection 1-forms* and view them as the entries of an $n \times n$ matrix Ω .

The calculation above applies in an arbitrary smooth manifold. Now we consider a Riemannian manifold and consider the connection 1-forms are associated to an orthonormal frame field defined on an open set in M . The next proposition shows that, in this situation, the covariant derivative is compatible if and only if its matrix of connection forms is anti-symmetric.

Proposition 3.3. *Let M be a Riemannian n -manifold. Let $\Omega = [\omega_{ij}]$ be the matrix of connection 1-forms determined by a covariant derivative ∇ and an orthonormal frame field*

(U_1, \dots, U_n) defined on an open subset of M . Then the covariant derivative ∇ is compatible with the metric on M if and only if $\omega_{ij} = -\omega_{ji}$ for all $i, j \in \{1, \dots, n\}$.

Proof. Assume that ∇ is compatible. Since $\langle U_i, U_j \rangle$ is constant we have, for any vector field X ,

$$0 = X \cdot \langle U_i, U_j \rangle = \langle \nabla_X(U_i), U_j \rangle + \langle U_i, \nabla_X(U_j) \rangle = \omega_{ij}(X) + \omega_{ji}(X).$$

For the converse, assume that $\omega_{ij} = -\omega_{ji}$ and let $X = \sum a_i U_i$, $Y = \sum b_i U_i$ and Z be vector fields. We have

$$Z(\langle X, Y \rangle) = \sum a_i db_i(Z) + b_i da_i(Z).$$

Applying 3.2.1 we have

$$\langle \nabla_Z(X), Y \rangle = \sum b_i da_i(Z) + \sum b_j a_i \omega_{ji}(Z);$$

and

$$\langle X, \nabla_Z(Y) \rangle = \sum a_i db_i(Z) + \sum a_j b_i \omega_{ij}(Z).$$

Thus we must show that

$$\sum b_j a_i \omega_{ij}(Z) + \sum a_j b_i \omega_{ij}(Z) = 0.$$

But if we interchange the names of the indices in the second sum the left hand side becomes

$$\sum b_j a_i \omega_{ij}(Z) + \sum a_i b_j \omega_{ji}(Z),$$

which is equal to 0 since we have assumed that $\omega_{ji}(Z) + \omega_{ij}(Z) = 0$. □

Remark 3.4. Assuming that an orthonormal frame field has been fixed, we have shown that a covariant derivative is completely described by the anti-symmetric matrix Ω of 1-forms. But the axioms of a covariant derivative are expressed without reference to any choice of frame field. One would therefore expect that there is an invariantly defined object which plays the same rôle as Ω . Indeed, such an object exists and is called a *connection*. Connections can be defined on arbitrary vector bundles. A connection which is defined on the tangent bundle is called an *affine connection*. One way to define a connection is as a Lie algebra valued 1-form. We have been looking at the case where the values lie in the Lie algebra of the orthogonal group. (When the orthogonal group is identified with the group of orthogonal matrices, its Lie algebra is identified with the algebra of anti-symmetric matrices.)

For these notes we will always work with a frame field, so we will not try to give the definition of a Lie Algebra valued 1-form. Look it up if you are interested.

4. Brackets

Let M be a general (i.e. non-Riemannian) smooth n -manifold. Recall that vector fields may be viewed as differential operators on smooth functions, i.e. as directional derivatives. As such there is a canonically defined operation defined on the $C^\infty(M)$ -module of smooth vector fields $\mathcal{X}(M)$, namely the operation of composition, as differential operators. If X and Y are two vector fields then we denote their composition as $X \circ Y$. Given a smooth function f we have

$$X \circ Y(f) = X(df(Y)) = d(df(Y))(X).$$

Note that, as with covariant differentiation, the composition operation is not a tensor; instead it satisfies a ‘‘Leibniz rule’’. It is true, for smooth functions f and g and smooth vector fields X, Y and Z , that

$$(fX + gY) \circ Z = fX \circ Z + gY \circ Z.$$

However the composition operation is not $C^\infty(M)$ -linear in the second variable. In fact, we have

$$X \circ (fY) = X(f)Y + fX \circ Y = df(X)Y + fX \circ Y.$$

Definition 4.1. Let X and Y be smooth vector fields defined on the smooth manifold M . The *Lie bracket* of X and Y is the smooth vector field given by

$$[X, Y] = X \circ Y - Y \circ X.$$

Note that $[X, Y] = -[Y, X]$ and $[fX, Y] = f[X, Y] - Y(f)X$. One can check that the bracket also satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0,$$

but we won't need that.

A key point about the bracket is that its definition does not refer to a coordinate system. Thus any quantity whose definition can be formulated in terms of covariant derivatives and brackets is automatically invariant under coordinate changes.

It is not difficult to express the bracket in terms of local coordinates; and we will only have to use this expression once.

Proposition 4.2. Let x_1, \dots, x_n be coordinate functions determined by a chart (U, ϕ) for a smooth n -manifold M . Suppose that $X = \sum a_i \frac{\partial}{\partial x_i}$ and $Y = \sum b_i \frac{\partial}{\partial x_i}$ are smooth vector fields defined on U . Then

$$[X, Y] = \sum (a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}) \frac{\partial}{\partial x_j}.$$

Proof. We have for a smooth function f ,

$$X \circ Y(f) = \sum a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

and

$$Y \circ X(f) = \sum b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} + \sum b_j a_i \frac{\partial^2 f}{\partial x_j \partial x_i}$$

When these expressions are subtracted the second order derivatives cancel and the lemma follows. \square

The bracket enables us to give a coordinate-free formula for the differential of a 1-form. This will allow us to make our computations directly in terms of frame fields, without having to represent them in terms of local coordinates.

Proposition 4.3. *Let ω be a 1-form and X and Y smooth vector fields on an n -manifold M . Then*

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

(Here $\omega(Y)$ and $\omega(X)$ are smooth functions and $X(\omega(Y))$ and $Y(\omega(X))$ are directional derivatives of these functions.)

Proof. Let x_1, \dots, x_n be local coordinates given by a chart (U, ϕ) . Write X, Y and ω in terms of these coordinates: $X = \sum a_i \frac{\partial}{\partial x_i}$, $Y = \sum b_j \frac{\partial}{\partial x_j}$ and $\omega = \sum z_i dx_i$.

According to Proposition 4.2 we have

$$\omega([X, Y]) = \sum (a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}) z_i.$$

Expanding the other two terms, we have

$$\begin{aligned} X(\omega(Y)) &= X(\sum b_j z_j) \\ &= \sum (db_j(X) z_j + b_j dz_j(X)) \\ &= \sum a_j z_j db_j(\frac{\partial}{\partial x_j} + \sum b_i dz_i(X)) \\ &= \sum a_j z_j \frac{\partial b_j}{\partial x_j} + \sum b_j dz_j(X) \end{aligned}$$

and

$$Y(\omega(X)) = \sum b_j z_j \frac{\partial a_i}{\partial x_j} + \sum a_i dz_i(X).$$

Thus

$$\begin{aligned} X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) &= \sum b_j dz_j(X) - \sum a_i dz_i(X) \\ &= \sum (dz_j(X) dx_i(Y) - dz_i(Y) dx_j(X)) \\ &= d\omega(X, Y) \end{aligned}$$

\square

5. Torsion

Definition 5.1. The *torsion* of a covariant derivative ∇ on a smooth manifold M is the map $T : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ given by

$$T(X, Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y].$$

Note that this definition is global, and does not require a choice of coordinates or a choice of frame field.

Since covariant differentiation and composition fail to be bilinear in exactly the same way, we have the following

Proposition 5.2. *The torsion of a covariant derivative on a smooth manifold M is an alternating bilinear map of $C^\infty(M)$ -modules.*

Proof. The alternating property is clear from the definition. To verify the bilinearity we compute:

$$\begin{aligned} T(X, fY) &= \nabla_X(fY) - X \circ (fY) + \nabla_{fY}(X) - fY \circ X \\ &= X(f)Y + f\nabla_X(Y) - X(f)Y - fX \circ Y + f\nabla_Y(X) - fY \circ X \\ &= fT(X, Y). \end{aligned}$$

□

This observation invites us to express T in terms of differential forms.

To a frame field (E_1, \dots, E_n) we associate the column vector Θ for which the i^{th} entry is the dual 1-form θ_i to E_i .

We may represent the vector field $T(X, Y)$ in terms of our frame field as a column vector whose entries are alternating bilinear functions of X and Y . Thus the frame field determines a representation of T as a column vector of 2-forms. We will denote this column vector as $[T]$.

It will be convenient to extend the usual matrix multiplication to matrices of k -forms by replacing the usual product with the wedge product. (Of course one must be careful to preserve the order of multiplication, since the wedge product is anti-commutative.) Given a matrix A of k -forms we define dA to be the matrix obtained by applying the operator d to each entry. It is easy to check that the product rule holds for matrix multiplication. If A is a matrix of p -forms and B is a matrix of q -forms and if the sizes of the matrices are compatible for matrix multiplication, then

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB.$$

It is important to note, though, that if A is a matrix of forms then it can easily happen that $A \wedge A \neq 0$.

Proposition 5.3. Let M be a smooth manifold with a covariant derivative ∇ . Let (E_1, \dots, E_n) be a frame field on an open subset of M with associated column vector Θ of dual 1-forms. Let Ω be the representation of ∇ as a matrix of connection forms, and $[T]$ the representation of the torsion T of ∇ as a column vector of 2-forms. Then

$$[T] = d\Theta - \Omega \wedge \Theta.$$

Proof. We will show, for each index i , that

$$d\theta_i(X, Y) - \theta_i(T(X, Y)) = \sum \omega_{ij} \wedge \theta_j(X, Y).$$

We write the vector fields in terms of our frame field: $X = \sum a_i E_i$ and $Y = \sum b_i E_i$. According to Proposition 4.3 we have

$$\begin{aligned} d\theta_i(X, Y) - \theta_i(T(X, Y)) &= X(b_i) - Y(a_i) - \theta_i([X, Y]) - \theta_i(\nabla_X(Y) - \nabla_Y(X) - [X, Y]) \\ &= db_i(X) - da_i(Y) - \theta_i(\nabla_X(Y)) + \theta_i(\nabla_Y(X)) \end{aligned}$$

Now, according to 3.2.1 we have

$$-da_i(Y) + \theta_i(\nabla_Y(X)) = a_j \omega_{ji}(Y)$$

and

$$db_i(X) - \theta_i(\nabla_X(Y)) = -b_j \omega_{ji}(X).$$

It follows that

$$\begin{aligned} d\theta_i(X, Y) - \theta_i(T(X, Y)) &= \omega_{ji}(X)a_j - \omega_{ji}(Y)b_j \\ &= \omega_{ij}(X)\theta_j(Y) - \omega_{ij}(Y)\theta_j(X) \\ &= \omega_{ij} \wedge \theta_j(X, Y). \end{aligned}$$

□

We are finally ready to describe the canonical compatible covariant derivative associated to a Riemannian metric.

We fix an orthonormal frame field (U_1, \dots, U_n) defined on an open set in a Riemannian n -manifold M . There is a dual 1-form θ_i associated to each of the vector fields U_i . In this case we have

$$\theta_i(X) = \langle X, U_i \rangle.$$

We will view the 1-forms θ_i as the entries of a column vector Θ .

Definition 5.4. A covariant derivative on a Riemannian manifold is *Levi-Civita* if it is compatible with the metric and has vanishing torsion. In particular, this means that $d\Theta = \Omega \wedge \Theta$.

Proposition 5.5. A Riemannian n -manifold has a unique Levi-Civita covariant derivative.

Proof. The idea is that the equation $d\Theta = \Omega \wedge \Theta$ completely determines Ω , given that Ω is anti-symmetric. We have, for each i and each $j \neq k$,

$$d\theta_i(U_k, U_j) = \sum \omega_{ij} \wedge \theta_j(U_k, U_j).$$

Thus $\omega_{ij}(U_k) = d\theta_i(U_k, U_j)$ for $j \neq k$. This determines the value of ω_{ij} for all but one of the vector fields in the frame field. For the last vector field, we use the fact that ω is anti-symmetric to see that $\omega_{ij}(U_j) = -\omega_{ji}(U_j) = d\theta_j(U_j, U_i)$. Thus the value of ω_{ij} on each vector field in the frame field is uniquely determined by $d\Theta$. \square

6. Curvature

Definition 6.1. Let M be a smooth n -manifold with a covariant derivative ∇ . The *curvature* of ∇ is the mapping $\mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ given by

$$(X, Y, Z) \mapsto R_{X,Y}(Z) = \nabla_X(\nabla_Y(Z)) - \nabla_Y(\nabla_X(Z)) - \nabla_{[X,Y]}(Z).$$

Note that this definition is global, and does not require a choice of coordinates or a frame field.

The next proposition says that the curvature is a tensor.

Proposition 6.2. *If M is a smooth n -manifold with a covariant derivative ∇ then the curvature of ∇ is a multi-linear map of $C^\infty(M)$ -modules.*

Proof. We have

$$\begin{aligned} \nabla_{fX}(\nabla_Y(Z)) &= f\nabla_X(\nabla_Y(Z)); \\ \nabla_Y(\nabla_{fX}(Z)) &= \nabla_Y(f\nabla_X(Z)) = Y(f)\nabla_X(Z) + f\nabla_Y(\nabla_X(Z)); \text{ and} \\ \nabla_{[fX,Y]}(Z) &= \nabla_{f[X,Y]-Y(f)X}(Z) = f\nabla_{[X,Y]}(Z) - Y(f)\nabla_X(Z). \end{aligned}$$

Summing these three terms, we obtain $R_{fX,Y}(Z) = fR_{X,Y}(Z)$. The other two computations are similar. \square

Now suppose that (E_1, \dots, E_n) is a frame field defined in an open subset W in M . The multilinearity implies that $R_{X,Y}(\cdot)$ restricts to a linear transformation from T_pM to itself for each $p \in W$. We will denote by $R_{X,Y}$ the family of linear transformations obtained in this way. We may use the frame field to write $R_{X,Y}$ as a matrix $[\rho_{ij}(X, Y)]$ of smooth functions on W . Clearly, $R_{X,Y}(Z) = -R_{Y,X}(Z)$. Moreover, since $R_{X,Y}(Z)$ is bilinear, the ρ_{ij} are in fact 2-forms. Thus a choice of frame field determines a representation of the curvature of ∇ as a matrix of 2-forms, which we will denote as $[R]$.

The next proposition relates this matrix of 2-forms to the connection 2-forms.

Proposition 6.3. Let M be a smooth manifold with a covariant derivative ∇ . Let (E_1, \dots, E_n) be a frame field defined on an open set in M , with dual 1-forms $\theta_1, \dots, \theta_n$. Let Θ be the column vector whose i^{th} entry is θ_i ; let Ω be the matrix of connection forms determined by the frame field; and let $[R]$ be the matrix of 2-forms which represents the curvature of ∇ . Then

$$[R] = d\Omega - \Omega \wedge \Omega.$$

Proof. Let $[R] = [\rho_{ij}]$ where the ρ_{ij} are 2-forms. We have $\rho_{ij} = \theta_i(R_{X,Y}(E_j))$. Thus we must show

$$\theta_i(R_{X,Y}(E_j)) = d\omega_{ij}(X, Y) - \sum \omega_{ik} \wedge \omega_{kj}(X, Y).$$

We have

$$\begin{aligned} R_{X,Y}(E_j) &= \nabla_X(\nabla_Y(E_j)) - \nabla_Y(\nabla_X(E_j)) - \nabla_{[X,Y]}(E_j) \\ &= \nabla_X(\sum \omega_{kj}(Y)E_k) - \nabla_Y(\sum \omega_{kj}(X)E_k) - \sum \omega_{kj}([X, Y])E_k \\ &= \sum X(\omega_{kj}(Y))E_k + \sum \omega_{kj}(Y)\nabla_X(E_k) \\ &\quad - \sum Y(\omega_{kj}(X))E_k - \sum \omega_{kj}(X)\nabla_Y(E_k) \\ &\quad - \sum \omega_{kj}([X, Y])E_k \end{aligned}$$

We now apply 4.3 to obtain

$$\begin{aligned} R_{X,Y}(E_j) &= \sum d\omega_{kj}(X, Y)E_k + \sum \omega_{kj}(Y)\nabla_X(E_k) - \sum \omega_{kj}(X)\nabla_Y(E_k) \\ &= \sum d\omega_{kj}(X, Y)E_k + \sum \omega_{kj}(Y)\omega_{lk}(X)E_l - \sum \omega_{kj}(X)\omega_{lk}(Y)E_l \\ &= \sum d\omega_{kj}(X, Y)E_k + \sum \omega_{lk} \wedge \omega_{kj}(X, Y)E_l. \end{aligned}$$

Finally, applying θ_i yields the required identity. \square

7. Surfaces

In this section we will describe the curvature of a smooth oriented Riemannian 2-manifold Σ . We will then restrict to the case of a surface embedded in \mathbb{R}^3 and interpret its curvature geometrically.

Choose a positive orthonormal frame field (U_1, U_2) defined on an open set in Σ . Let Θ be the column vector of dual 1-forms and let Ω be the matrix of connection forms for the Levi-Civita covariant derivative ∇ on Σ

We observe that, even though the 1-forms θ_i obviously depend on our choice of a positive orthonormal frame field, the 2-form $\theta_1 \wedge \theta_2$ does not. If (U'_1, U'_2) is another positive orthonormal frame field and if A_p denotes the unique orthogonal linear transformation carrying $(U_1(p), U_2(p))$ to $(U'_1(p), U'_2(p))$ then on $T_p\Sigma$ we have

$$\theta'_1 \wedge \theta'_2 = (\det A_p)\theta_1 \wedge \theta_2 = \theta_1 \wedge \theta_2.$$

Since this 2-form is invariantly defined on any chart neighborhood, it is globally defined on Σ .

Definition 7.1. The globally defined 2-form which agrees with $\theta_1 \wedge \theta_2$ is called *the area form* of Σ and will be denoted dA . If Φ is a compact 2-dimensional manifold-with-boundary contained in Σ then we define the *area* of Φ to be $\int_{\Phi} dA$.

If (X, Y) is any positive frame field defined on an open set W in Σ then we have $dA(X, Y) > 0$ at each point of W . In particular, a compact submanifold-with-boundary has positive area. In addition, this implies that any 2-form defined on W can be written as $f\theta_1 \wedge \theta_2$ for some smooth function f .

For any point p in $\Sigma \cap W$ let $J_p : T_p\Sigma \rightarrow T_p\Sigma$ denote the linear transformation such that $J_p(U_1) = U_2$ and $J_p(U_2) = -U_1$. Thus J_p is a 90° rotation in the counter-clockwise direction. It is clear from this description (or from an easy computation) that J_p commutes with any orthogonal linear transformation from T_p to T_p . Thus if (U'_1, U'_2) is another positive orthonormal frame field, and if $J'_p : T_p\Sigma \rightarrow T_p\Sigma$ denotes the linear transformation such that $J'_p(U'_1) = U'_2$ and $J'_p(U'_2) = -U'_1$ then we have $J_p = J'_p$. That is, there is a well-defined map $J : T\Sigma \rightarrow T\Sigma$ that preserves each tangent space $T_p\Sigma$ and acts on that oriented vector space as a counter-clockwise rotation by 90° .

Definition 7.2. The *almost-complex structure* on Σ is the unique map $J : T\Sigma \rightarrow T\Sigma$ with the following property. If (U_1, U_2) is a positive orthonormal frame field defined on an open set in Σ then $J(U_1) = U_2$ and $J(U_2) = -U_1$.

Next we consider the matrix Ω . Since the covariant derivative ∇ is compatible, Ω is anti-symmetric. Hence it can be written as

$$\Omega' = \begin{bmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{bmatrix}.$$

It follows immediately that $\Omega \wedge \Omega = 0$ and therefore Proposition 6.3 implies that

$$[R] = \Omega \wedge \Omega - d\Omega = \begin{bmatrix} 0 & -d\omega_{12} \\ d\omega_{12} & 0 \end{bmatrix}.$$

We have seen that there exists a smooth function κ which is defined on W and satisfies $-d\omega_{12} = \kappa\theta_1 \wedge \theta_2$. The computation above shows that $R(U_1, U_2) = -\kappa J$. It follows from the fact that $R(X, Y)$ is alternating and bilinear that the linear transformation $R(U_1, U_2)$ is equal to $R(U'_1, U'_2)$ for any other positive orthonormal frame field (U'_1, U'_2) . This implies that the definition of κ is independent of the choice of a positive orthonormal frame field, and hence that κ is a globally defined smooth function on Σ .

Definition 7.3. The *Gauss curvature* of Σ is the unique smooth function κ defined globally on Σ and having the following property. Let (U_1, U_2) be any positive orthonormal frame field defined on an open subset of Σ , let Θ be the column vector of dual 1-forms and let Ω the

matrix of connection 1-forms for the Levi-Civita covariant derivative on Σ . Then

$$-d\omega_{12} = \kappa\theta_1 \wedge \theta_2.$$

8. Surfaces in \mathbb{R}^3

We now restrict to the situation where Σ is a 2-manifold embedded in \mathbb{R}^3 . The orientation of Σ corresponds to a choice of a normal vector field defined on Σ . We choose an extension of this normal vector field to a vector field N defined in a neighborhood of Σ .

We work with the standard Riemannian metric on \mathbb{R}^3 , and define the Riemannian metric on Σ to be the restriction of the standard metric on \mathbb{R}^3 to the tangent bundle of Σ . It is clear that the Levi-Civita covariant derivative associated to the standard metric on \mathbb{R}^3 is just the standard covariant derivative D . The vector fields $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_3})$ form an orthonormal frame field on \mathbb{R}^3 and it is easy to check in terms of this frame field that the curvature of \mathbb{R}^3 is 0.

Now suppose that X and Y are vector fields on Σ . We define a covariant derivative on Σ by $\nabla_X(Y) = D_X(Y) - \langle D_X(Y), N \rangle$, i.e. by orthogonally projecting the standard covariant derivative to the tangent space of Σ . (Strictly speaking, this involves extending X and Y to a neighborhood of Σ and then observing that the definition does not depend on the choice of extension.) Throughout this discussion we will need to extend or restrict vector fields and forms to or from a neighborhood. To simplify notation we will not distinguish a vector field or form from its extension or restriction. The choice of extension will not affect any computation that we make on Σ .

Fix an orthonormal frame field (U_1, U_2, U_3) defined in an \mathbb{R}^3 -neighborhood of an open subset W of Σ , chosen so that $U_3 = N$ and (U_1, U_2) is a positive frame field on Σ . As usual, let Θ be the column vector of dual 1-forms and let Ω be the associated 3×3 matrix of connection forms.

We regard (U_1, U_2) as an orthonormal frame field on Σ . The associated column vector Θ' of dual 1-forms thus consists of the first two entries of Θ . The definition of ∇ implies that the 2×2 matrix Ω' of connection forms for this frame field on Σ consists of the first two rows and columns of Ω .

Differentiating both sides of the equation $U_i \cdot U_i = 1$ shows that, for any vector field X we have $D_X(U_i) \cdot U_i = 0$. It follows that if X is a vector field which is tangent to Σ then $D_X(U_3)$ is another vector field tangent to Σ .

Definition 8.1. The *shape operator* (or Weingarten map) of the oriented embedded surface $\Sigma \subset \mathbb{R}^3$ is the map $S : \mathcal{X}(\Sigma) \rightarrow \mathcal{X}(\Sigma)$ given by

$$S(X) = -D_X(N)$$

where N is the unit normal vector field to Σ determined by its orientation. The restriction of S to a tangent plane $T_p\Sigma$ is a linear endomorphism of $T_p\Sigma$ and will be denoted S_p

The link between the shape operator and the geometry of the surface S is provided by the next proposition. Given a point $p \in \Sigma$ and a unit tangent vector $v \in T_p\Sigma$, the intersection of Σ with the plane spanned by $N(p)$ and v will be called the *normal section* through v .

Proposition 8.2. *Let $p \in \Sigma$ and let $v \in T_p\Sigma$. Then the curvature at p of the normal section through the unit tangent vector v is given by $S_p(v) \cdot v$.*

Proof. Let $\sigma(s)$ be a unit speed parametrization of the normal section, with $\sigma(0) = p$ and $\sigma'(0) = v$. The unit normal vector to the section is $N(\sigma(t))$. Thus the curvature of the section is $\sigma'' \cdot N$. Since $\sigma' \cdot N = 0$, taking covariant derivatives gives

$$\nabla_{\sigma'}(\sigma') \cdot N + \sigma' \cdot \nabla_{\sigma'}(N) = 0.$$

The left hand side of this equation is the curvature of the section, while the right hand side is equal to $S(v) \cdot v$. \square

We may consider the matrix representation of S_p with respect to the orthonormal basis $(U_1(p), U_2(p))$. Letting p vary we obtain a matrix of smooth functions which we compute in terms of the connection forms as:

$$[S] = \begin{bmatrix} -\nabla_{U_1}(U_3) \cdot U_1 & -\nabla_{U_2}(U_3) \cdot U_1 \\ -\nabla_{U_1}(U_3) \cdot U_2 & -\nabla_{U_2}(U_3) \cdot U_2 \end{bmatrix} = \begin{bmatrix} -\omega_{13}(U_1) & -\omega_{13}(U_2) \\ -\omega_{23}(U_1) & -\omega_{23}(U_2) \end{bmatrix} = \begin{bmatrix} \omega_{31}(U_1) & \omega_{31}(U_2) \\ \omega_{32}(U_1) & \omega_{32}(U_2) \end{bmatrix}$$

The next proposition shows that $S_p(v) \cdot v$ is a quadratic form.

Proposition 8.3. *The linear transformation S_p is symmetric.*

Proof. It suffices to show that the matrix $[S]$ is symmetric, i.e. that $\omega_{31}(U_2) = \omega_{32}(U_1)$. Looking at the bottom row of the equation $d\Theta = \Omega \wedge \Theta$ we see that

$$d\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2.$$

Thus we have

$$d\theta_3(\theta_1, \theta_2) = -\omega_{31}(U_2) - \omega_{32}(U_1).$$

Therefore it suffices to show that $d\theta_3(\theta_1, \theta_2) = 0$. We may use 4.3 to compute

$$d\theta_3(\theta_1, \theta_2) = U_1(\theta_3(U_2)) - U_2(\theta_3(U_1)) - \theta_3([U_1, U_2]).$$

Since θ_3 vanishes on each tangent space to Σ , the right hand side is 0. This completes the proof. \square

ADD DISCUSSION OF THE FUNDAMENTAL FORMS HERE.

Finally we connect the shape operator to the Gauss curvature.

Proposition 8.4. *The determinant of S_p is the Gauss curvature of Σ at p .*

Proof. Taking the determinant of $[S]$ we obtain

$$\omega_{31}(U_1)\omega_{32}(U_2) - \omega_{31}(U_2)\omega_{32}(U_1) = \omega_{31} \wedge \omega_{32}(U_1, U_2).$$

Next we observe that, since the curvature of \mathbb{R}^3 is 0, we have $d\Omega = \Omega \wedge \Omega$. Expanding the 12-entry of both sides we see

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{31} \wedge \omega_{32}.$$

Since $\kappa = -d\omega_{12}$, the result follows. \square

Since S_p is symmetric, it has an orthonormal basis of eigenvectors. The function $S_p(v) \cdot v$ on the circle of unit vectors in $T_p\Sigma$ attains its extreme values at the unit vectors in the eigenspaces. Consequently the two eigendirections of S_p in $T_p\Sigma$, which are called the *principal directions*, are the directions in which the curvature of the normal section is maximized or minimized. The curvatures in these two directions are called the *principal curvatures*. Gauss originally defined the curvature of Σ at a point p to be the product of the principal curvatures at p . Gauss' Theorema Egregium states that the curvature is determined by the intrinsic metric on Σ . We have just verified this: the product of the principal curvatures is determined by the 1-form ω_{12} , which in turn depends only on the Riemannian metric of Σ .

9. Computations

In this section we compute the Gauss curvature of some familiar surfaces.

Example 9.1. The *hyperbolic upper half-plane* \mathbb{H}^2 is the Riemannian manifold with underlying space $\{(x, y) | y > 0\}$ endowed with the Riemannian metric given, for vector fields X and Y , by

$$\langle X, Y \rangle = \frac{1}{y^2} X \cdot Y,$$

where $X \cdot Y$ denotes the standard Riemannian metric on \mathbb{R}^2 . Thus if $p = (x, y) \in \mathbb{H}^2$ then the hyperbolic length of a tangent vector in $T_p\mathbb{H}^2$ is equal to its Euclidean length divided by y .

Consider the orthonormal frame field (U_1, U_2) given by

$$U_i(x, y) = y \frac{\partial}{\partial x_i}, \text{ for } i = 1, 2.$$

The dual 1-forms to this frame field are $\theta_1 = \frac{1}{y} dx$ and $\theta_2 = \frac{1}{y} dy$. We compute the connection form ω_{12} by solving the equation $d\Theta = \Omega \wedge \Theta$. We have

$$\begin{aligned} d\theta_1 &= \frac{1}{y^2} dx \wedge dy = \theta_1 \wedge \theta_2; \\ d\theta_2 &= 0 \end{aligned}$$

Thus

$$\omega_{12} \wedge \theta_2 = \theta_1 \wedge \theta_2;$$

$$\omega_{12} \wedge \theta_1 = 0$$

which implies that $\omega_{12} = \theta_1$ and hence that $d\omega_{12} = \theta_1 \wedge \theta_2$.

Since the Gauss curvature κ is defined by $d\omega_{12} = -\kappa\theta_1 \wedge \theta_2$, we have shown that \mathbb{H}^2 has curvature -1 at each point.

Example 9.2. Next we consider the sphere of radius R in \mathbb{R}^3 . Since it is obvious that the isometry group of the sphere acts transitively, it suffices to compute the Gauss curvature at one point. Actually, we will use an orthonormal frame field that is defined everywhere on the sphere except at the north and south poles.

For the orthonormal frame field we take U_1 to be a unit vector pointing south and U_2 to be a unit vector pointing east. We can easily describe the dual 1-forms in terms of the usual spherical coordinates (ρ, ϕ, θ) on \mathbb{R}^3 . To avoid confusion with the spherical coordinate, we will let η_1 and η_2 denote the dual 1-forms.

We have

$$\eta_1 = R d\phi;$$

$$\eta_2 = R \sin \phi d\theta$$

In particular, this implies $\eta_1 \wedge \eta_2 = R^2 \sin \phi d\phi \wedge d\theta$.

We solve for the connection 1-form ω_{12} using the equation $d\eta = \Omega \wedge \eta$:

$$\omega_{12} \wedge \eta_2 = d\eta_1 = 0$$

$$\omega_{12} \wedge \eta_1 = -d\eta_2 = -R \cos \phi d\phi \wedge d\theta.$$

Thus $\omega_{12}(U_1) = 0$ and

$$\omega_{12}(U_2) = \mathbb{R} \cos \phi d\phi \wedge d\theta(U_1, U_2) = \frac{R \cos \phi}{R^2 \sin \phi}.$$

It follows that

$$\omega_{12} = \frac{\cos \phi}{R \sin \phi} \eta_2.$$

Differentiating, we have

$$\begin{aligned} d\omega_{12} &= d\left(\frac{\cos \phi}{R \sin \phi}\right) \wedge \eta_2 + \frac{\cos \phi}{R \sin \phi} d\eta_2 \\ &= \frac{-1}{R \sin^2 \phi} d\phi \wedge \eta_2 + \frac{\cos^2 \phi}{R \sin \phi} d\phi \wedge d\theta \\ &= \frac{-1}{R^2 \sin^2 \phi} d\eta_1 \wedge \eta_2 + \frac{\cos^2 \phi}{R^2 \sin^2 \phi} d\eta_1 \wedge d\eta_1 \\ &= \frac{-1}{R^2} d\eta_1 \wedge d\eta_2. \end{aligned}$$

Since $d\omega_{12} = -\kappa\eta_1 \wedge \eta_2$, we conclude that $\kappa = \frac{1}{R^2}$ for a sphere of radius R .