Math 121 – Exam 2 Solutions

1. (12 pts) Complete each of the following.

   (a) Find the **EXACT** value of $\log_2 16$.

   (b) Find the **EXACT** value of $3^{2\log_3 4}$.

   (c) If $u = \ln 2$ and $v = \ln 3$, write the following expression in terms of u and v:

$$\ln 8 + \ln 9 - \ln 12$$

   (d) Write the following expression as a single logarithm.

$$\log_2 x^2 - \log_2(x+2) + 3\log_2(x-1)$$

Solution:

(a) $\log_2 16 = \log_2 2^4 = 4\log_2 2 = \boxed{4}$

(b) $3^{2\log_3 4} = 3^{\log_3 4^2} = 3^{\log_3 16} = \boxed{16}$

(c) $\ln 8 + \ln 9 - \ln 12 = \ln\left(\frac{8\cdot 9}{12}\right) = \ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3 = \boxed{u+v}$

(d) $\log_2 x^2 - \log_2(x+2) + 3\log_2(x-1) = \boxed{\log_2\left[\frac{x^2(x-1)^3}{x+2}\right]}$

2. (15 pts) Find all possible solutions to the following equation:

$$\log_2 x + \log_2(x-8) = 3$$

Solution:

$$\begin{aligned} \log_2 x + \log_2(x-8) &= 3 \\ \log_2[x(x-8)] &= 3 \\ x(x-8) &= 2^3 \\ x^2 - 8x &= 8 \\ x^2 - 8x - 8 &= 0 \end{aligned}$$

Using the quadratic equation to solve for x we get:

$$\begin{aligned} x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-8)}}{2(1)} \\ x &= \frac{8 \pm \sqrt{64+32}}{2} \\ x &= \frac{8 \pm \sqrt{96}}{2} \\ x &= \frac{8 \pm 4\sqrt{6}}{2} \\ x &= 4 \pm 2\sqrt{6} \end{aligned}$$

We eliminate the negative root since $\log_2(4-2\sqrt{6})$ does not exist. Therefore, the answer is $\boxed{x = 4 + 2\sqrt{6}}$.

3. (15 pts) Find all possible solutions to the following equation:

$$(e^4)^x \cdot e^{x^2} = e^{12}$$

Solution:

$$\begin{aligned} (e^4)^x \cdot e^{x^2} &= e^{12} \\ e^{4x} e^{x^2} &= e^{12} \\ e^{4x+x^2} &= e^{12} \\ 4x + x^2 &= 12 \\ x^2 + 4x - 12 &= 0 \\ (x+6)(x-2) &= 0 \\ x = -6,\ x &= 2 \end{aligned}$$

Both are solutions since the domain of the exponential function is all real numbers. Therefore, $\boxed{x = -6,\ 2}$.

4. (16 pts) A detective is called to investigate the scene of a crime where a dead body has been found. She measures the body temperature to be 80° F at 10:09 PM. The thermostat in the room where the body lies reads 68° F. The temperature of the body is taken exactly 1 hour later and is found to be 78° F. Use Newton's Law of Cooling (t measured in hours) to answer the following questions.

$$u(t) = T + (u_0 - T)e^{kt}, \qquad k < 0$$

(a) Determine the decay constant k and write your answer to three decimal places.

(b) Using the value of k found in part (a), estimate the time of death assuming that the victim's body temperature was 98.6° F prior to death. Write your answer in the form `xx:xx` PM.
(Hint: Find the value of t such that $u(t) = 98.6°$ F. The value of t you get should be negative. Then backtrack from 10:09 PM to figure out the time of death.)

Solution:

(a) Here we have $T = 68$ and $u_0 = 80$. Using the fact that $u(1) = 78$, we solve for k:

$$\begin{aligned} u(t) &= T + (u_0 - T)e^{kt} \\ u(1) &= 68 + (80 - 68)e^{k(1)} \\ 78 &= 68 + 12e^k \\ 10 &= 12e^k \\ \frac{10}{12} &= e^k \\ e^k &= \frac{5}{6} \\ k &= \ln\frac{5}{6} \approx \boxed{-0.182} \end{aligned}$$

(b) To find the time of death, we find the value of t such that $u(t) = 98.6$:

$$\begin{aligned} 98.6 &= 68 + (80 - 68)e^{-0.182t} \\ 30.6 &= 12e^{-0.182t} \\ e^{-0.182t} &= \frac{30.6}{12} \\ -0.182t &= \ln \frac{30.6}{12} \\ t &= -\frac{\ln \frac{30.6}{12}}{0.182} \approx 5.14 \end{aligned}$$

5.14 hours is the equivalent of 5 hours, 9 minutes (to the nearest minute). Therefore, the time of death is approximately $\boxed{\text{5:00 PM}}$.

5. (12 pts) Find the **EXACT** values of the following expressions:

(a) $\cos 30°$ (b) $\tan \frac{3\pi}{4}$ (c) $\sec 240°$ (d) $\csc \frac{5\pi}{6}$

Solution:

(a) $\cos 30° = \frac{\sqrt{3}}{2}$

(b) $\tan \frac{3\pi}{4} = -1$

(c) $\sec 240° = -2$

(d) $\csc \frac{5\pi}{6} = 2$

6. (15 pts) Given that $\tan\theta = \frac{1}{2}$ and $\sin\theta < 0$, compute $\cos\theta$, $\sin(-\theta)$, and $\cot\theta$.

Solution: Since $\tan\theta > 0$ and $\sin\theta < 0$, we must have $\cos\theta < 0$. Then use an identity to compute $\sec\theta$:

$$\begin{aligned} \tan^2\theta + 1 &= \sec^2\theta \\ \left(\frac{1}{2}\right)^2 + 1 &= \sec^2\theta \\ \frac{1}{4} + 1 &= \sec^2\theta \\ \sec^2\theta &= \frac{5}{4} \\ \sec\theta &= -\frac{\sqrt{5}}{2} \end{aligned}$$

We used the negative root above since $\cos\theta < 0 \Rightarrow \sec\theta < 0$. Therefore,

$$\cos\theta = \frac{1}{\sec\theta} = \boxed{-\frac{2}{\sqrt{5}}}$$

Using another identity, we find $\sin\theta$:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \sin^2\theta + \left(-\frac{2}{\sqrt{5}}\right)^2 &= 1\\ \sin^2\theta + \frac{4}{5} &= 1\\ \sin^2\theta &= \frac{1}{5}\\ \sin\theta &= -\frac{1}{\sqrt{5}}\end{aligned}$$

Therefore,

$$\begin{aligned}\sin(-\theta) = -\sin\theta &= \boxed{\frac{1}{\sqrt{5}}}\\ \cot\theta = \frac{1}{\tan\theta} &= \boxed{2}\end{aligned}$$

7. (15 pts) Find values of A, ω, and ϕ such that the graph of $y = A\sin(\omega x - \phi)$ has the following properties:

$$\text{amplitude} = 2, \qquad \text{period} = 4, \qquad \text{phase shift} = 1$$

Then plot **one cycle** of the graph on the grid below. (Each square in the grid has a side of length 1.)

Solution: Since the amplitude is 2, we take $A = 2$. Since the period is 4 we have:

$$\begin{aligned}\text{period} &= \frac{2\pi}{\omega}\\ 4 &= \frac{2\pi}{\omega}\\ \omega &= \frac{\pi}{2}\end{aligned}$$

Since the phase shift is 1 we have:

$$\begin{aligned}\text{phase shift} &= \frac{\phi}{\omega}\\ 1 &= \frac{\phi}{\frac{\pi}{2}}\\ \phi &= \frac{\pi}{2}\end{aligned}$$

The function is then $\boxed{y = 2\sin\left(\frac{\pi}{2}x - \frac{\pi}{2}\right)}$.