Math 121 – Fall 2007
Sample Exam 3 Solutions

1. (20 pts) Compute the **EXACT** values of:

   (a) $\sin 45°$ (b) $\sin 22.5°$ (c) $\cos\left(\frac{\pi}{12}\right)$

   Solution:

   (a) $\sin 45° = \boxed{\frac{\sqrt{2}}{2}}$

   (b) Use the half-angle identity:

$$\sin 22.5° = \sin\frac{45°}{2} = +\sqrt{\frac{1-\cos 45°}{2}} = \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

   (c) Use either the half-angle identity or the subtraction identity for cos:

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi/6}{2}\right) = +\sqrt{\frac{1+\cos(\pi/6)}{2}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

   or

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right)\\ &= \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}\end{aligned}$$

   Note: $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

2. (20 pts) The terminal side of an angle t passes through the point $(-1,-2)$. Compute the **EXACT** values of:

   (a) $\cos t$ (b) $\sin t$ (c) $\sin 2t$

   Solution:

   (a) Since $x=-1$ and $y=-2$ we have:

$$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$$

   Then, by definition:

$$\cos t = \frac{x}{r} = \boxed{-\frac{1}{\sqrt{5}}}$$

(b) Also, by definition:

$$\sin t = \frac{y}{r} = \boxed{-\frac{2}{\sqrt{5}}}$$

(c) Using the double angle formula for sin we have:

$$\sin 2t = 2\sin t\cos t = 2\left(-\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}\right) = \boxed{\frac{4}{5}}$$

3. (15 pts) If $\csc x = 2$ and $\frac{\pi}{2} \leq x \leq \pi$, find:

   (a) $\cos x$

   (b) $\sin\left(x + \frac{\pi}{3}\right)$

Solution:

(a) We first obtain $\sin x$ by using the definition for $\csc x$:

$$\begin{aligned} \csc x &= 2 \\ \frac{1}{\sin x} &= 2 \\ \sin x &= \frac{1}{2} \end{aligned}$$

Since $\sin x = \frac{1}{2}$ and $\frac{\pi}{2} \leq x \leq \pi$, we have $x = 150^\circ = \frac{5\pi}{6}$. Thus,

$$\cos x = \boxed{-\frac{\sqrt{3}}{2}}$$

(b) Since $x = \frac{5\pi}{6}$, $x + \frac{\pi}{3} = \frac{5\pi}{6} + \frac{\pi}{3} = \frac{7\pi}{6}$. Thus,

$$\sin\left(x + \frac{\pi}{3}\right) = \sin\frac{7\pi}{6} = \boxed{-\frac{1}{2}}$$

Or, we can use the addition formula for sin we have:

$$\begin{aligned} \sin\left(x + \frac{\pi}{3}\right) &= \sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3} \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \boxed{-\frac{1}{2}} \end{aligned}$$

4. (15 pts) Prove the following identity:

$$\cos 3x = 4\cos^3 x - 3\cos x$$

Solution: Use the addition formula for cos:

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= (\cos^2 x - \sin^2 x)\cos x - (2\sin x\cos x)\sin x\\ &= \cos^3 x - \sin^2 x\cos x - 2\sin^2 x\cos x\\ &= \cos^3 x - 3\sin^2 x\cos x\\ &= \cos^3 x - 3(1-\cos^2 x)\cos x\\ &= \cos^3 x - 3\cos x + 3\cos^3 x\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

5. (15 pts) Compute the **EXACT** value of:

$$\sin\left(\cos^{-1}\left(-\frac{1}{4}\right)\right)$$

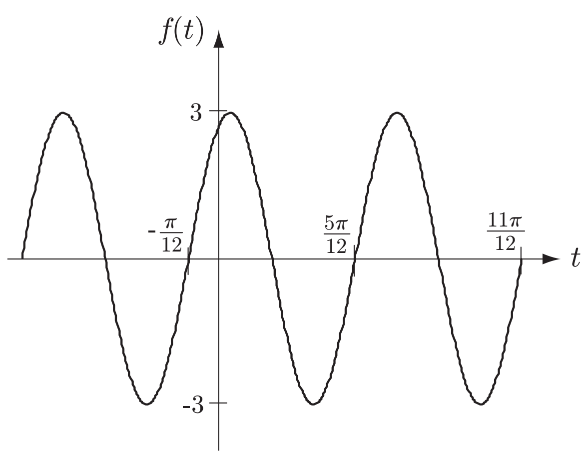
Solution: Let $u = \cos^{-1}\left(-\frac{1}{4}\right)$, where $0 \le u \le \pi$. Then $\cos u = -\frac{1}{4}$. Using the Pythagorean Identity, we have:

$$\begin{aligned}\sin^2 u + \cos^2 u &= 1\\ \sin^2 u + \left(-\frac{1}{4}\right)^2 &= 1\\ \sin^2 u &= \frac{15}{16}\\ \sin u &= \pm\frac{\sqrt{15}}{4}\end{aligned}$$

Since $0 \le u \le \pi$, $\sin u = \frac{\sqrt{15}}{4}$. Therefore,

$$\sin\left(\cos^{-1}\left(-\frac{1}{4}\right)\right) = \sin u = \boxed{\frac{\sqrt{15}}{4}}$$

6. (15 pts) Consider the graph of a function $f(t)$ below.



(a) Determine values of A, b, and c such that $f(t) = A\sin(bt + c)$.

(b) Find the period, amplitude, and phase shift of $f(t)$.

Solution: From the graph, we know that:

$$\begin{aligned} \text{amplitude} &= 3 \\ \text{period} &= \frac{5\pi}{12} - \left(-\frac{\pi}{12}\right) = \frac{\pi}{2} \\ \text{phase shift} &= -\frac{\pi}{12} \end{aligned}$$

We let $\boxed{A = 3}$. Then, by definition:

$$\text{period} = \frac{2\pi}{b} \quad \Rightarrow \quad \frac{\pi}{2} = \frac{2\pi}{b} \quad \Rightarrow \quad \boxed{b = 4}$$

Finally, by definition:

$$\text{phase shift} = -\frac{c}{b} \quad \Rightarrow \quad -\frac{\pi}{12} = -\frac{c}{4} \quad \Rightarrow \quad \boxed{c = \frac{\pi}{3}}$$