Math 121 – Exam 2 Solutions

1. (15 pts) Find all roots (real and complex) of the polynomial function:

$$f(x) = x^3 + x^2 + 9x + 9$$

Clearly show how you obtained your answers.

Solution: First, we look for the rational roots. The factors of the constant term $a_0 = 9$ are: $\pm 1, \pm 3, \pm 9$. The factors of the leading coefficient $a_3 = 1$ are: ± 1. Therefore, the possible rational roots are: $\pm 1, \pm 3, \pm 9$. Of these, only -1 is a root since:

$$f(-1) = (-1)^3 + (-1)^2 + 9(-1) + 9 = 0$$

This root could also have been found by using the calculator or by guessing.

Then, to find the remaining roots, we divide $f(x)$ by the $x + 1$ (the factor corresponding to the root -1):

$$\begin{array}{r|l} & x^2 \qquad + 9 \\ \hline x+1 & x^3 + x^2 + 9x + 9 \\ & -x^3 - x^2 \\ \hline & \qquad\qquad 9x + 9 \\ & \qquad\qquad -9x - 9 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

Taking the quotient, $x^2 + 9$, and setting it equal to zero we get:

$$\begin{aligned} x^2 + 9 &= 0 \\ x^2 &= -9 \\ x &= \pm\sqrt{-9} \\ x &= \pm 3i \end{aligned}$$

Therefore, the roots are $\boxed{-1, 3i, -3i}$.

Note: The problem could also have been solved by directly factoring $f(x)$ as follows:

$$\begin{aligned} f(x) &= x^3 + x^2 + 9x + 9 \\ f(x) &= x^2(x + 1) + 9(x + 1) \\ f(x) &= (x + 1)(x^2 + 9) \end{aligned}$$

Setting each of the factors equal to zero gives us the roots we found above.

2. (20 pts) For the rational function $f(x) = \dfrac{(x+1)(x-1)^2}{x(x+2)(x-1)^2}$ determine:

   (a) the roots of $f(x)$,
   (b) the vertical asymptotes,
   (c) the holes (if any),
   (d) the horizontal asymptote (if there is one).

Solution: This problem was on Quiz 7.

   (a) The only root is $\boxed{x = -1}$ (this is a root of the numerator).
   (b) The vertical asymptotes are $\boxed{x = 0}$ and $\boxed{x = -2}$ (these are roots of the denominator).
   (c) There is a hole at $\boxed{x = 1}$ (the multiplicity of 1 as a root of the numerator is the same as that of the denominator).
   (d) The horizontal asymptote is $\boxed{y = 0}$ (the degree of the numerator, 3, is less than the degree of the denominator, 4).

3. (15 pts) Solve the inequality: $x^2 + 4x + 3 \leq 0$.

Solution: First, factor the left hand side:

$$f(x) = x^2 + 4x + 3 \leq 0$$
$$f(x) = (x+3)(x+1) \leq 0$$

The roots of $f(x)$ are $x = -3$ and $x = -1$. To find the x-values that satisfy the inequality, we construct a table:

interval:	$x \leq -3$	$-3 \leq x \leq -1$	$x \geq -1$
test number, c:	-4	-2	0
$f(c)$:	3	-1	3
sign of $f(c)$:	$+$	$-$	$+$

Since we have $f(x) \leq 0$, we choose the "$-$" interval: $\boxed{-3 \leq x \leq -1}$.

4. (15 pts) Simplify the following expressions:

(a) $\dfrac{2}{i+1}$ (write in the form $a+bi$)

(b) $\left(\dfrac{a^{1/2}}{b}\right)^2 \sqrt[3]{a^6b^2}$ (write in the form $a^x b^y$)

(c) $2\ln x - \ln(x+1) + 1$ (write as a single logarithm – hint: $v = \ln e^v$)

Solution:

(a) $\dfrac{2}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{2-2i}{1-i^2} = \dfrac{2-2i}{1+1} = \dfrac{2-2i}{2} = \boxed{1-i}$

(b) $\left(\dfrac{a^{1/2}}{b}\right)^2 \sqrt[3]{a^6b^2} = ab^{-2}a^2b^{2/3} = \boxed{a^3b^{-4/3}}$

(c) $2\ln x - \ln(x+1) + 1 = \ln x^2 - \ln(x+1) + \ln e = \boxed{\ln\left(\dfrac{ex^2}{x+1}\right)}$

5. (20 pts) Suppose you put \$5000 in a bank account where the interest is compounded monthly.

   (a) How long will it take for the account to reach \$6000 if the interest rate is 4%? Write your answer in years and to 2 decimal places.

   (b) At what interest rate would it take 4 years for the account to reach \$6000? Write your answer as a percentage and to 2 decimal places.

Hint: $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Solution:

(a) Plugging in $P = 5000$, $A(t) = 6000$, $r = 0.04$, $n = 12$ (monthly), and solving for t we get:

$$\begin{aligned}
6000 &= 5000\left(1 + \frac{0.04}{12}\right)^{12t} \\
\frac{6000}{5000} &= (1 + 0.00\bar{3})^{12t} \\
1.2 &= (1.00\bar{3})^{12t} \\
\ln 1.2 &= \ln(1.00\bar{3})^{12t} \\
\ln 1.2 &= (12t)\ln 1.00\bar{3} \\
12t &= \frac{\ln 1.2}{\ln 1.00\bar{3}} \\
t &= \frac{1}{12}\frac{\ln 1.2}{\ln 1.00\bar{3}} \\
&\boxed{t = 4.57 \text{ years}}
\end{aligned}$$

(b) Plugging in $P = 5000$, $A(t) = 6000$, $t = 4$, $n = 12$ (monthly), and solving for r we have:

$$\begin{aligned}
6000 &= 5000\left(1 + \frac{r}{12}\right)^{12(4)} \\
\frac{6000}{5000} &= \left(1 + \frac{r}{12}\right)^{48} \\
1.2 &= \left(1 + \frac{r}{12}\right)^{48} \\
(1.2)^{1/48} &= \left[\left(1 + \frac{r}{12}\right)^{48}\right]^{1/48} \\
(1.2)^{1/48} &= 1 + \frac{r}{12} \\
r &= 12(1.2^{1/48} - 1) \\
r &= 0.0457
\end{aligned}$$

Therefore, the interest rate is $\boxed{r = 4.57\%}$.

6. (15 pts) Find all solutions to the equation: $\ln x + \ln(x+1) = \ln 3 + \ln 4$.

Solution: Solving, we have:

$$\begin{aligned} \ln x + \ln(x+1) &= \ln 3 + \ln 4 \\ \ln(x(x+1)) &= \ln(3 \cdot 4) \\ \ln(x^2 + x) &= \ln 12 \\ e^{\ln(x^2+x)} &= e^{\ln 12} \\ x^2 + x &= 12 \\ x^2 + x - 12 &= 0 \\ (x+4)(x-3) &= 0 \\ x &= -4,\ 3 \end{aligned}$$

However, $x = -4$ does not work because if we plug it back into the equation we get $\ln(-3) + \ln(-4)$ on the left hand side and we cannot take the natural logarithm of a negative number. So, the answer is $\boxed{x = 3}$.