Math 121 – Exam 3 Solutions

1. (20 pts) Compute the **EXACT** values of:

   (a) $\cos\frac{3\pi}{4}$ (b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (c) $\tan\frac{\pi}{6}$ (d) $\csc\frac{5\pi}{6}$ (e) $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

   Solution:

   (a) $\cos\frac{3\pi}{4} = \boxed{-\frac{\sqrt{2}}{2}}$

   (b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{-\frac{\pi}{3}}$

   (c) $\tan\frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{3}}$

   (d) $\csc\frac{5\pi}{6} = \boxed{2}$

   (e) $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$

2. (20 pts) If $\cos x = \frac{2}{3}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$, then find the **EXACT** values of:

   (a) $\sin x$ (b) $\cos 2x$ (c) $\sin(x+\pi)$

   Solution:

   (a) The terminal side of x lies in Quadrant IV, where $\sin x < 0$ and $\cos x > 0$. Using the Pythagorean Identity, we have:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\sin^2 x + \left(\frac{2}{3}\right)^2 &= 1 \\
\sin^2 x + \frac{4}{9} &= 1 \\
\sin^2 x &= \frac{5}{9} \\
\boxed{\sin x = -\frac{\sqrt{5}}{3}}
\end{aligned}$$

   (b) Using the double angle identity, we have:

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
&= \left(\frac{2}{3}\right)^2 - \left(-\frac{\sqrt{5}}{3}\right)^2 \\
&= \frac{4}{9} - \frac{5}{9} \\
&= \boxed{-\frac{1}{9}}
\end{aligned}$$

(c) Using the addition identity, we have:

$$\begin{aligned}
\sin(x+\pi) &= \sin x \cos \pi + \cos x \sin \pi \\
&= \left(-\frac{\sqrt{5}}{3}\right)(-1) + \left(\frac{2}{3}\right)(1) \\
&= \boxed{\frac{\sqrt{5}}{3}}
\end{aligned}$$

3. (15 pts) A function $f(t)$ is known to have the following properties:

$$\text{amplitude} = \pi, \ \text{period} = 2, \ \text{phase shift} = -1$$

Find the values of A, b, and c such that $f(t) = A\cos(bt + c)$.

Solution: Since amplitude $= |A| = \pi$, we have $\boxed{A = 2}$ (or we could have $A = -2$). Using the definition of the period, we have:

$$\begin{aligned}
\text{period} = \frac{2\pi}{b} &= 2 \\
b &= \frac{2\pi}{2} \\
&\boxed{b = \pi}
\end{aligned}$$

Using the definition of the phase shift, we have:

$$\begin{aligned}
\text{phase shift} = -\frac{c}{b} &= -1 \\
-\frac{c}{\pi} &= -1 \\
&\boxed{c = \pi}
\end{aligned}$$

4. (15 pts) Prove the following identity:

$$\sec x - \cos x = \sin x \tan x$$

Solution: To prove the identity, we will start with the left side and convert it:

$$\begin{aligned}
\sec x - \cos x &= \frac{1}{\cos x} - \cos x \\
&= \frac{1 - \cos^2 x}{\cos x} \\
&= \frac{\sin^2 x}{\cos x} \\
&= \sin x \cdot \frac{\sin x}{\cos x} \\
&= \sin x \tan x
\end{aligned}$$

5. (15 pts) Find the **EXACT** values of the following expressions:

(a) $\cot\left(\sin^{-1}\dfrac{2}{5}\right)$ (b) $\cos^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

Solution:

(a) Let $x = \sin^{-1}\dfrac{2}{5}$. Then $\sin x = \dfrac{2}{5}$ and the terminal side of x is in Quadrant I. Using the Pythagorean Identity, we have:

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1\\ \left(\frac{2}{5}\right)^2 + \cos^2 x &= 1\\ \frac{4}{25} + \cos^2 x &= 1\\ \cos^2 x &= \frac{21}{25}\\ \cos x &= \frac{\sqrt{21}}{5}\end{aligned}$$

Thus,

$$\begin{aligned}\cot\left(\sin^{-1}\frac{2}{5}\right) &= \cot x\\ &= \frac{\cos x}{\sin x}\\ &= \boxed{\frac{\sqrt{21}}{2}}\end{aligned}$$

(b) $\cos^{-1}\left(\tan\dfrac{3\pi}{4}\right) = \cos^{-1}(-1) = \boxed{\pi}$

6. (15 pts) Find all solutions to the equation:

$$\sin 3x = \frac{1}{2}$$

in the interval $0 \leq x \leq 5$.

Solution: Let $\theta = 2x$. One solution to $\sin\theta = \dfrac{1}{2}$ is $\theta = \dfrac{\pi}{6}$. All solutions are:

$$\begin{aligned}\theta &= \frac{\pi}{6} + 2k\pi, \qquad \text{where } k = 0, \pm1, \pm2, \ldots\\ \theta &= \left(\pi - \frac{\pi}{6}\right) + 2k\pi\\ &= \frac{5\pi}{6} + 2k\pi\end{aligned}$$

Substituting $3x$ for θ, we have:

$$\begin{aligned}3x &= \frac{\pi}{6} + 2k\pi\\ 3x &= \frac{5\pi}{6} + 2k\pi\\ \Rightarrow \quad x &= \frac{\pi}{18} + \frac{2k\pi}{3}, \qquad \text{where } k = 0, \pm1, \pm2, \ldots\\ x &= \frac{5\pi}{18} + \frac{2k\pi}{3}\end{aligned}$$

The solutions that lie in the interval $0 \leq x \leq 5$ are:

$$\boxed{x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}}$$