MATH 220 **FINAL EXAM** **12.10.98 - 6pm**

Directions: Answer all questions and show all work in the **exam booklet** provided. Start each new question at the **top** of a new page and **box** your final answer. Each of the 8 questions is worth 25 points. There are two Bonus questions worth 10pts each.

1. (a) Find $y(x)$: $\dfrac{dy}{dx} = 4\sqrt{y+4}\ \sin x \cos x, \quad y(\pi/2) = 0.$

   (b) Find the general solution $y(x)$: $y'' = \dfrac{1}{x}y' - \dfrac{4}{x^2}y, \quad x > 0.$

2. Find the general solution $y(x)$ of:

$$y'' - y = 4e^x - 10\sin 2x$$

   Suggestion: Use the method of undetermined coefficients.

3. A thin, insulated wire of length 50 has its temperature at the end $x = 0$ fixed at 5 and at the end $x = 50$ fixed at 45 . The diffusivity of the wire is $\beta = 3$. Initially, the temperature is 0 for $0 < x < 20$ and 10 for $20 \leq x < 50$. Let $u(x, t)$ be the temperature in the wire at position x and time t.

   (a) State the heat conduction problem for $u(x, t)$, i.e. PDE+BC+IC.

   (b) Find the steady-state solution $v(x)$, i.e. the time independent, long time behavior of $u(x, t)$.

   (c) **Bonus (10pts):** Describe the method and state the problem (with homogeneous boundary conditions) that can be used to construct the full time varying solution $u(x, t)$.

4. (a) Construct the Fourier series for the function

$$f(x) = \begin{cases} 2, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$$

   (b) Sketch a graph showing the values to which this series converges on $-2\pi < x < 2\pi$.

5. Find the *smallest* real value of the constant $\lambda \geq 0$ (eigenvalue) for which the boundary value problem

$$y'' + 2\lambda y' + 5\lambda^2 y = 0, \quad 0 < x < \pi; \quad y(0) = 0, \quad y(\pi) = 0$$

has a nontrivial solution. Also determine the corresponding solution (eigenfunction).

6. (a) Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$:

$$y'' + 6y' = t^2 e^{-5t} + e^{-t}\cos 2t, \quad y(0) = 2, \ \ y'(0) = 0$$

(b) Find $\mathcal{L}\{h(t) + \delta(t-\pi)\cos t\}$ if $h(t) = \begin{cases} 0, & 0 < t < 3 \\ 5, & 3 < t < 6 \\ 0, & t > 6. \end{cases}$

(c) Compute the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\} =$$

7. Find the solution $x(t)$ of the system of equations:

$$\begin{aligned} x' + y &= 1 - u(t-2), \ \ x(0) = 0 \\ x + y' &= 0, \quad y(0) = 0 \end{aligned}$$

8. A 1000L transfer tank (well-stirred) initially contains 100L of brine solution with a total amount of 5 kg of salt. Solution leaves the tank from the bottom at the rate of 2 L/min and fluid enters the tank at the top at the rate of 4 L/min. The incoming fluid has a salt concentration of 2 kg/L.

(a) At what time will the tank overflow?

(b) Find $x(t)$ which is the amount of salt in the tank at time t.

(c) At time $t = 50$, a supervisor notices that the tank will eventually overflow and increases the rate at which the fluid leaves the bottom of the tank to 4 L/min. Compute $x(t)$ for $t > 50$. Hint: Use your result in part(b) for $0 \leq t \leq 50$.

Bonus-10pts: Find a particular solution to $x^2 y'' - 2xy' + 2y = x^2 \ln x, \ x > 0$.