Math 220 FINAL EXAM 05/10/07

Directions: Answer all questions and show all (intermediate) work in the booklet provided. Start each new question at the top of a new page and $\boxed{\text{box}}$ your final answer.

1. (a) (15 pts) Solve the initial value problem:

$$(x^2+1)\frac{dy}{dx} = x - xy\,, \quad y(0) = -1.$$

   (b) (10 pts) Consider the initial value problem:

$$\frac{dx}{dt} = x^2 + 2t,\ x(1) = 2.$$

   Use Euler's method with step size $h = \frac{1}{2}$ to estimate $x(2)$.

2. (20 pts) A tank has a capacity of 20 L. It initially contains 10 L of pure water. A solution of salt and water is fed into the tank at a rate of 2 L/min with a concentration of 1 kg/L. The solution is drained from the tank at a rate of 1 L/min.

   (a) At what time T will the tank be filled?

   (b) Determine $x(t)$, the amount of salt in the tank at time t.

3. (20 pts) Find the general solutions to the ordinary differential equations:

   (a) $x^2y'' - xy' + y = 0$

   (b) $y'' - 3y' + 16y = 0$

4. (25 pts) Consider the equation $y'' + 4y' + 4y = e^{-2t} - 3$.

   (a) Find the homogeneous solution.

   (b) Find a particular solution (**you must solve for the unknown constants**).

   (c) Write down the general solution.

5. (15 pts) Find the solutions $x(t)$ and $y(t)$ to the following system of ODEs:

$$y' - 2y = x, \quad x(0) = 1$$
$$x' + y' = 0, \quad y(0) = 1$$

6. (30 pts) Evaluate the following expressions:

   (a) $\mathscr{L}[(t+1)^3 e^t + e^{-2t}\sin 3t]$

   (b) $\mathscr{L}^{-1}\left[\frac{s+6}{s^2+4s+20} + \frac{G(s)}{s^2}\right]$ where $G(s) = \mathscr{L}[g(t)]$

7. (25 pts) Consider the following initial value problem:

$$x'' - 4x = 3\delta(t-2),\ x(0) = 0,\ x'(0) = 1$$

(a) Find the Laplace transform of the solution $X(s)$.

(b) Use the inverse Laplace transform to find the solution $x(t)$.

8. (20 pts) Consider the function $f(x) = \begin{cases} 1, & \text{if } -2\pi < x < -\pi \\ 0, & \text{if } -\pi < x < \pi \\ 1, & \text{if } \pi < x < 2\pi \end{cases}$.

(a) Compute the Fourier expansion of $f(x)$ on $[-2\pi, 2\pi]$.

(b) Define the function to which the Fourier series from part (a) converges.

9. (20 pts) Find the solution $u(x,t)$ to:

$$\begin{aligned} u_t &= u_{xx} \\ \text{I.C.}:\ u(x,0) &= \pi + \cos 2x - 3\cos 5x \\ \text{B.C.s}:\ \frac{\partial u}{\partial x}(0,t) &= 0 = \frac{\partial u}{\partial x}(\pi,t) \end{aligned}$$