Spring, 2007 – Final Exam Solutions

1. (a) (15 pts) Solve the initial value problem: $(x^2+1)\dfrac{dy}{dx} = x - xy\,, \quad y(0) = -1$

   (b) (10 pts) Consider the initial value problem: $\dfrac{dx}{dt} = x^2 + 2t,\ x(1) = 2$. Use Euler's method with step size $h = \dfrac{1}{2}$ to estimate $x(2)$.

Solution:

(a) This equation is separable:

$$
\begin{aligned}
(x^2+1)\frac{dy}{dx} &= x - xy \\
(x^2+1)\frac{dy}{dx} &= x(1-y) \\
\frac{1}{1-y}dy &= \frac{x}{x^2+1}dx \\
\int \frac{1}{1-y}dy &= \int \frac{x}{x^2+1}dx \\
-\ln|1-y| &= \frac{1}{2}\ln|x^2+1| + C
\end{aligned}
$$

Plugging in the initial condition $y(0) = -1$ to solve for C we get:

$$
\begin{aligned}
-\ln|1-(-1)| &= \frac{1}{2}\ln|0^2+1| + C \\
-\ln 2 &= C
\end{aligned}
$$

The solution is then:

$$\boxed{-\ln|1-y| = \frac{1}{2}\ln|x^2+1| - \ln 2}$$

(b) Here, we have $t_0 = 1$ and $x_0 = 2$. One iteration of Euler's Method gives us:

$$
\begin{aligned}
x_1 &= x_0 + hf(t_0, x_0) = 2 + \frac{1}{2}(2^2 + 2(1)) = 5 \\
t_1 &= t_0 + h = 1 + \frac{1}{2} = \frac{3}{2}
\end{aligned}
$$

A second iteration gives us:

$$
\begin{aligned}
x_2 &= x_1 + hf(t_1, x_1) = 5 + \frac{1}{2}\left[5^2 + 2\left(\frac{3}{2}\right)\right] = 19 \\
t_2 &= t_1 + h = \frac{3}{2} + \frac{1}{2} = 1
\end{aligned}
$$

Therefore, $\boxed{x(2) \approx 19}$.

2. (20 pts) A tank has a capacity of 20 L. It initially contains 10 L of pure water. A solution of salt and water is fed into the tank at a rate of 2 L/min with a concentration of 1 kg/L. The solution is drained from the tank at a rate of 1 L/min.

   (a) At what time T will the tank be filled?
   (b) Determine $x(t)$, the amount of salt in the tank at time t.

Solution:

(a) The rate at which solution is being fed into the tank is greater than the rate at which it released from it. Therefore, the volume will increase according to the equation:

$$V(t) = V_0 + (r_i - r_o)t = 10 + (2-1)t = 10 + t$$

The time T when the tank is filled is then:

$$20 = 10 + T$$
$$\boxed{T = 10 \text{ min}}$$

(b) The ODE governing the amount of salt in the tank at time t is:

$$\begin{aligned}
\frac{dx}{dt} &= \text{rate in} - \text{rate out} \\
\frac{dx}{dt} &= r_i c_i - r_o c_o \\
\frac{dx}{dt} &= (2)(1) - (1)\frac{x}{V(t)} \\
\frac{dx}{dt} &= 2 - \frac{x}{10+t} \\
\frac{dx}{dt} + \frac{1}{10+t} &= 2
\end{aligned}$$

The equation is linear but not separable. We must use the integrating factor:

$$\mu(t) = \exp\left(\int \frac{1}{10+t}\right) = e^{\ln(10+t)} = 10 + t$$

The solution is then:

$$\begin{aligned}
x(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)q(t)\,dt + C\right) \\
x(t) &= \frac{1}{10+t}\left(\int (10+t)2\,dt + C\right) \\
x(t) &= \frac{1}{10+t}\left(20t + t^2 + C\right)
\end{aligned}$$

Initially, the tank contains pure water. So, $x(0) = 0$. We use this initial condition to find C:

$$\begin{aligned}
x(0) = \frac{1}{10+0}\left(20(0) + 0^2 + C\right) &= 0 \\
\frac{C}{10} &= 0 \\
C &= 0
\end{aligned}$$

The solution is then:

$$\boxed{x(t) = \frac{20t + t^2}{10 + t}}$$

3. (20 pts) Find the general solutions to the ordinary differential equations:

(a) $x^2y'' - xy' + y = 0$

(b) $y'' - 3y' + 16y = 0$

Solution:

(a) This is a Cauchy-Euler equation. The indicial equation and its roots are:

$$\begin{aligned} r(r-1) - r + 1 &= 0 \\ r^2 - 2r + 1 &= 0 \\ (r-1)^2 &= 0 \\ r &= 1 \end{aligned}$$

Since the root $r = 1$ is repeated, the general solution is:

$$\boxed{y(x) = c_1 x + c_2 x \ln x}$$

(b) The auxiliary and its roots are:

$$\begin{aligned} r^2 - 3r + 16 = 0 \\ r &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(16)}}{2(1)} \\ r &= \frac{3 \pm \sqrt{9 - 64}}{2} \\ r &= \frac{3}{2} \pm i\frac{\sqrt{55}}{2} \end{aligned}$$

The roots are complex conjugates. Therefore, the general solution is:

$$\boxed{y(x) = e^{3x/2}\left[c_1 \cos\left(\frac{\sqrt{55}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{55}}{2}x\right)\right]}$$

4. (25 pts) Consider the equation $y'' + 4y' + 4y = e^{-2t} - 3$.

   (a) Find the homogeneous solution.

   (b) Find a particular solution (**you must solve for the unknown constants**).

   (c) Write down the general solution.

Solution:

(a) The auxiliary equation and its roots are:

$$\begin{aligned} r^2 + 4r + 4 &= 0 \\ (r+2)^2 &= 0 \\ r &= -2 \end{aligned}$$

Since the root $r = -2$ is repeated, the homogeneous solution is:

$$\boxed{y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}}$$

(b) The form of the particular solution (because of the repeated root) is:

$$y_p(t) = At^2 e^{-2t} + B$$

Its derivatives are:

$$\begin{aligned} y_p'(t) &= 2Ate^{-2t} - 2At^2e^{-2t} \\ y_p''(t) &= 2Ae^{-2t} - 4Ate^{-2t} - 4Ate^{-2t} + 4At^2e^{-2t} = 2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t} \end{aligned}$$

Plugging these expressions into the ODE we get:

$$\begin{aligned} y_p'' + 4y_p' + 4y_p &= e^{-2t} - 3 \\ 2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t} + 4(2Ate^{-2t} - 2At^2e^{-2t}) + 4(At^2e^{-2t} + B) &= e^{-2t} - 3 \\ 2Ae^{-2t} + 4B &= e^{-2t} - 3 \end{aligned}$$

Therefore, we must have $2A = 1 \ \Rightarrow \ A = \frac{1}{2}$ and $4B = -3 \ \Rightarrow \ B = -\frac{3}{4}$. The particular solution is:

$$\boxed{y_p(t) = \frac{1}{2}t^2 e^{-2t} - \frac{3}{4}}$$

(c) The total solution is:

$$\boxed{y(t) = y_h(t) + y_p(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{2}t^2 e^{-2t} - \frac{3}{4}}$$

5. (20 pts) Find the solutions $x(t)$ and $y(t)$ to the following system of ODEs:

$$\begin{aligned} y' - 3y &= x, \quad x(0) = 0 \\ x' + y' &= 3y, \quad y(0) = 1 \end{aligned}$$

Solution: We'll use the elimination method to solve the system. We'll take the first equation (which is already solved for x in terms of y and y') and plug it into the second equation:

$$\begin{aligned} (y' - 2y)' + y' &= 0 \\ y'' - 2y' + y' &= 0 \\ y'' - y' &= 0 \end{aligned}$$

The auxiliary equation is $r^2 - r = 0$ and its roots are $r = 0, 1$. Therefore, the general solution for $y(t)$ is:

$$y(t) = c_1 + c_2 e^t$$

We get $x(t)$ from the first equation above:

$$\begin{aligned} x(t) &= y'(t) - 2y(t) \\ x(t) &= c_2 e^t - 2(c_1 + c_2 e^t) \\ x(t) &= -2c_1 - c_2 e^t \end{aligned}$$

We get c_1 and c_2 from the initial conditions:

$$\begin{aligned} x(0) = -2c_1 - c_2 &= 1 \\ y(0) = c_1 + c_2 &= 1 \end{aligned}$$

The solution to this system is $c_1 = -2$ and $c_2 = 3$. Therefore, the solutions to the system are:

$$\boxed{x(t) = 4 - 3e^t, \; y(t) = -2 + 3e^t}$$

6. (30 pts) Evaluate the following expressions:

(a) $\mathscr{L}[(t+1)^3e^t + e^{-2t}\sin 3t]$

(b) $\mathscr{L}^{-1}\left[\dfrac{s+6}{s^2+4s+20} + \dfrac{G(s)}{s^2}\right]$ where $G(s) = \mathscr{L}[g(t)]$

Solution:

(a) Evaluating the first term we get:

$$\begin{aligned}\mathscr{L}[(t+1)^3e^t] &= \mathscr{L}[t^3e^t + 3t^2e^t + 3te^t + e^t] \\ &= \frac{6}{(s-1)^4} + \frac{6}{(s-1)^3} + \frac{3}{(s-1)^2} + \frac{1}{s-1}\end{aligned}$$

Evaluating the second term we get:

$$\mathscr{L}[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}$$

The sum of the two terms is:

$$\boxed{\frac{6}{(s-1)^4} + \frac{6}{(s-1)^3} + \frac{3}{(s-1)^2} + \frac{1}{s-1} + \frac{3}{(s+2)^2+9}}$$

(b) To evaluate the first term, we must first complete the square in the denominator and then separate into two terms:

$$\begin{aligned}\frac{s+6}{s^2+4s+20} &= \frac{s+6}{(s+2)^2+20-4} \\ &= \frac{s+6}{(s+2)^2+16} \\ &= \frac{s+2}{(s+2)^2+16} + \frac{4}{(s+2)^2+16}\end{aligned}$$

Taking the inverse Laplace transform of the above expression we get:

$$\mathscr{L}^{-1}\left[\frac{s+2}{(s+2)^2+16} + \frac{4}{(s+2)^2+16}\right] = e^{-2t}\cos 4t + e^{-2t}\sin 4t$$

We must use convolution to evaluate the second term. First, we let $F(s) = \frac{1}{s^2}$. We then have $f(t) = \mathscr{L}^{-1}[F(s)] = t$. Next, we use convolution:

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{G(s)}{s^2}\right] &= f(t) * g(t) \\ &= t * g(t) \\ &= \int_0^t (t-v)g(v)\,dv\end{aligned}$$

The sum of the two terms is:

$$\boxed{e^{-2t}\cos 4t + e^{-2t}\sin 4t + \int_0^t (t-v)g(v)\,dv}$$

7. (20 pts) Consider the following initial value problem:

$$x'' - 4x = 3\delta(t-2),\ x(0) = 0,\ x'(0) = 1$$

(a) Find the Laplace transform of the solution $X(s)$.

(b) Use the inverse Laplace transform to find the solution $x(t)$.

Solution:

(a) Taking the Laplace transform of the equation we get:

$$\begin{aligned} \mathscr{L}[x''] - 4\mathscr{L}[x] &= 3\mathscr{L}[\delta(t-2)] \\ s^2X(s) - sx(0) - x'(0) - 4X(s) &= 3e^{-2s} \\ (s^2-4)X(s) - 1 &= 3e^{-2s} \\ \boxed{X(s) = \frac{1}{s^2-4} + \frac{3e^{-2s}}{s^2-4}} \end{aligned}$$

(b) To find $x(t)$ we take the inverse Laplace transform of $X(s)$ from part (a). First, though, we must perform partial fraction decomposition:

$$\begin{aligned} \frac{1}{s^2-4} &= \frac{A}{s+2} + \frac{B}{s-2} \\ 1 &= A(s-2) + B(s+2) \end{aligned}$$

Plugging in $s = 2$ we get $1 = 4B \Rightarrow B = \frac{1}{4}$. Plugging in $s = -2$ we get $1 = -4A \Rightarrow A = -\frac{1}{4}$. Therefore, the solution $x(t)$ is:

$$x(t) = \mathscr{L}^{-1}\left[-\frac{\frac{1}{4}}{s+2} + \frac{\frac{1}{4}}{s-2} - \frac{\frac{3}{4}e^{-2s}}{s+2} + \frac{\frac{3}{4}e^{-2s}}{s-2}\right]$$

$$\boxed{x(t) = -\frac{1}{4}e^{-2t} + \frac{1}{4}e^{2t} + \left(-\frac{3}{4}e^{-2(t-2)} + \frac{3}{4}e^{2(t-2)}\right)u(t-2)}$$

8. (20 pts) Consider the function $f(x) = \begin{cases} 1, & \text{if } -2\pi < x < -\pi \\ 0, & \text{if } -\pi < x < \pi \\ 1, & \text{if } \pi < x < 2\pi \end{cases}$.

(a) Compute the Fourier expansion of $f(x)$ on $[-2\pi, 2\pi]$.

(b) Define the function to which the Fourier series from part (a) converges.

Solution:

(a) The function is even so the Fourier expansion of $f(x)$ is the Cosine Series which means $b_n = 0$ for $n = 1, 2, \ldots$ The other Fourier coefficients are:

$$\begin{aligned} a_0 &= \frac{2}{2\pi} \int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi} \int_\pi^{2\pi} 1\,dx \\ &= \frac{1}{\pi}[\pi] \\ &= 1 \\ a_n &= \frac{2}{2\pi} \int_0^{2\pi} f(x) \cos \frac{n\pi x}{2\pi}\,dx \\ &= \frac{1}{\pi} \int_\pi^{2\pi} \cos \frac{nx}{2}\,dx \\ &= \frac{1}{\pi} \left[\frac{2}{n} \sin \frac{nx}{2} \right]_\pi^{2\pi} \\ &= \frac{2}{n\pi} \left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right) \right] \\ &= \frac{2}{n\pi} \left[0 - \sin\left(\frac{n\pi}{2}\right) \right] \\ &= -\frac{2}{n\pi} \sin \frac{n\pi}{2} \end{aligned}$$

The Fourier Series for $f(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{T}$$

$$f(x) = \frac{2}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2}$$

$$\boxed{f(x) = 1 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{(2k-1)\pi} \cos \frac{(2k-1)\pi x}{2}}$$

(b) The Fourier Series converges to 1 for $-2\pi \leq x < -\pi$, $\frac{1}{2}$ for $x = -\pi$, 0 for $-\pi < x < \pi$, $\frac{1}{2}$ for $x = \pi$, and 1 for $\pi < x < 2\pi$.

9. (20 pts) Find the solution $u(x,t)$ to:

$$u_t = u_{xx}$$
$$\text{I.C.}: \; u(x,0) = \pi + \cos 2x - 3\cos 5x$$
$$\text{B.C.s}: \; \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t)$$

Solution: The insulated boundary conditions tell us that the form of the solution is:

$$\begin{aligned} u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta n^2 \pi^2 t/L^2} \cos\frac{n\pi x}{L} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx \end{aligned}$$

From the initial condition we have:

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \pi + \cos 2x - 3\cos 5x$$

The coefficients a_n are:

$$a_0 = 2\pi, \; a_2 = 1, \; a_5 = -3; \; a_n = 0 \text{ for } n = 1,3,4,6,7,8,\ldots$$

The solution is then:

$$\boxed{u(x,t) = \pi + e^{-4t}\cos 2x - 3e^{-25t}\cos 5x}$$