Math 220 Quiz 1 Solution

1. Show that $\phi(x) = \sin x$ is a solution to the ODE:

$$y'' + y = 0$$

2. Consider the initial value problem:

$$x\frac{dy}{dx} + y = x^2, \quad y(1) = 0$$

   (a) Find the exact solution $y(x)$.

   (b) Use Euler's Method with step size $h = \frac{1}{2}$ to estimate $y(2)$.

Solution:

1. To show that $\phi(x) = \sin x$ is a solution, we plug it in for y in the ODE:

$$\begin{aligned} y'' + y &\stackrel{?}{=} 0 \\ \frac{d}{dx}\left(\frac{d}{dx}\sin x\right) + \sin x &\stackrel{?}{=} 0 \\ \frac{d}{dx}(\cos x) + \sin x &\stackrel{?}{=} 0 \\ -\sin x + \sin x &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

2. (a) The ODE is first order and linear but not separable. It's not necessary here to compute the integrating factor because we notice that the right hand side can already be written as $x\frac{dy}{dx} + y = \frac{d}{dx}[xy]$. (Note: If you divided the equation by x first and then computed the integrating factor, you would find that $\mu(x) = x$.) Therefore, we have:

$$\begin{aligned} x\frac{dy}{dx} + y &= x^2 \\ \frac{d}{dx}[xy] &= x^2 \\ \int d[xy] &= \int x^2\,dx \\ xy &= \frac{1}{3}x^2 + C \end{aligned}$$

Use $y(1) = 0$ to find C:

$$\begin{aligned} (1)(0) &= \frac{1}{3}(1)^3 + C \\ C &= -\frac{1}{3} \end{aligned}$$

The solution is then:

$$\boxed{y(x) = \frac{1}{3}x - \frac{1}{3x}}$$

(b) Using Euler's Method we have $h = \frac{1}{2}$, $y_0 = 0$, $x_0 = 1$, and $f(x, y) = \frac{x^2 - y}{x}$. One step gives us:

$$\begin{aligned}
y_1 &= y_0 + hf(x_0, y_0) \\
&= 0 + \frac{1}{2}\left(\frac{1^2 - 0}{1}\right) \\
&= 0 + \frac{1}{2} \\
&= \frac{1}{2} \\
x_1 &= x_0 + h \\
&= 1 + \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}$$

Another step gives us:

$$\begin{aligned}
y_2 &= y_1 + hf(x_1, y_1) \\
&= \frac{1}{2} + \frac{1}{2}\left(\frac{\left(\frac{3}{2}\right)^2 - \frac{1}{2}}{\frac{3}{2}}\right) \\
&= \frac{1}{2} + \frac{1}{2}\left(\frac{7}{6}\right) \\
&= \frac{13}{12} \\
x_2 &= x_1 + h \\
&= \frac{3}{2} + \frac{1}{2} \\
&= 2
\end{aligned}$$

Therefore, the approximate value of $y(2)$ is $\boxed{\frac{13}{12}}$.