Math 220 Quiz 2 Solution

1. A tank initially contains 50 L of pure water. A solution of salt and water flows into the tank at a rate of 2 L/min and has a concentration of 1 kg/L. The solution flows out of the tank at a rate of 2 L/min. Determine $x(t)$, the amount of salt in the tank at time t.

2. Write the general solution $y(x)$ to the following ODE:
$$y'' + y' - 2y = 0$$

3. Write the form of the particular solution to the ODE:
$$y'' + y' - 2y = 3e^{2x} + 4\sin x$$
Note: **DO NOT SOLVE FOR THE COEFFICIENTS!**

Solution:

1. The flow rates in and out are the same so the volume will remain constant at $V = 50$ L. The governing ODE is:
$$\begin{aligned}\frac{dx}{dt} &= \text{rate in} - \text{rate out}\\ \frac{dx}{dt} &= r_i c_i - r_o c_o\\ \frac{dx}{dt} &= (2)(1) - (2)\left(\frac{x}{V}\right)\\ \frac{dx}{dt} &= 2 - \frac{2}{50}x\\ \frac{dx}{dt} &= \frac{50-x}{25}\end{aligned}$$
The above equation is both separable and linear. We'll solve by separation of variables:
$$\begin{aligned}\int \frac{dx}{50-x} &= \int 25\,dt\\ -\ln|50-x| &= 25t + C\end{aligned}$$
Initially, there is pure water so $x(0) = 0$. Plugging in $t = 0$ and $x = 0$ and solving for C we get:
$$-\ln|50-0| = 25(0) + C \quad \Rightarrow \quad C = -\ln 50$$
The solution is:
$$\begin{aligned}-\ln|50-x| &= 25t - \ln 50\\ \ln|50-x| - \ln 50 &= -25t\\ \ln\left|\frac{50-x}{50}\right| &= -25t\\ \frac{50-x}{50} &= e^{-25t}\\ \boxed{x(t) = 50 - 50e^{-25t}}\end{aligned}$$

2. The auxiliary equation is $r^2 + r - 2 = (r+2)(r-1) = 0$. The roots are $r = -2, 1$. Thus, the general solution is:
$$\boxed{y(x) = c_1 e^{-2x} + c_2 e^x}$$

3. The form of the particular solution is:
$$\boxed{y_p(x) = Ae^{2x} + B\sin x + C\cos x}$$