Math 220 Quiz 2 Solution

1. A tank initially contains 20 L of pure water. A solution of salt and water flows into the tank at a rate of 3 L/min and has a concentration of 1 kg/L. The solution flows out of the tank at a rate of 3 L/min. Determine $x(t)$, the amount of salt in the tank at time t.

2. Write the general solution $y(x)$ to the following ODE:

$$y'' + 6y' + 8y = 0$$

3. Write the form of the particular solution to the ODE:

$$y'' + 6y' + 8y = x^2 - e^{2x}$$

Note: **DO NOT SOLVE FOR THE COEFFICIENTS!**

Solution:

1. The flow rates in and out are the same so the volume will remain constant at $V = 20$ L. The governing ODE is:

$$\begin{aligned}
\frac{dx}{dt} &= \text{rate in} - \text{rate out} \\
\frac{dx}{dt} &= r_i c_i - r_o c_o \\
\frac{dx}{dt} &= (3)(1) - (3)\left(\frac{x}{V}\right) \\
\frac{dx}{dt} &= 3 - \frac{3}{20}x \\
\frac{dx}{dt} &= \frac{60 - x}{20}
\end{aligned}$$

The above equation is both separable and linear. We'll solve by separation of variables:

$$\begin{aligned}
\int \frac{dx}{60 - x} &= \int 20\,dt \\
-\ln|60 - x| &= 20t + C
\end{aligned}$$

Initially, there is pure water so $x(0) = 0$. Plugging in $t = 0$ and $x = 0$ and solving for C we get:

$$-\ln|60 - 0| = 20(0) + C \quad \Rightarrow \quad C = -\ln 60$$

The solution is:

$$\begin{aligned}
-\ln|60 - x| &= 20t - \ln 60 \\
\ln|60 - x| - \ln 60 &= -20t \\
\ln\left|\frac{60 - x}{60}\right| &= -20t \\
\frac{60 - x}{60} &= e^{-20t} \\
\boxed{x(t) = 60 - 60e^{-20t}}
\end{aligned}$$

2. The auxiliary equation is $r^2 + 6r + 8 = (r + 4)(r + 2) = 0$. The roots are $r = -4, -2$. Thus, the general solution is:

$$\boxed{y(x) = c_1 e^{-4x} + c_2 e^{-2x}}$$

3. The form of the particular solution is:

$$\boxed{y_p(x) = A_2 x^2 + A_1 x + A_0 + Be^{2x}}$$