Math 220 Quiz 3 Solution

1. Find the general solution to the following system of first order ODEs:

$$x' + y' = 2$$
$$x' - y = x$$

2. Use eigenvalues/eigenvectors to find the general solution to the following system of first order ODEs:

$$\frac{d\overrightarrow{\mathbf{x}}}{dt} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \overrightarrow{\mathbf{x}}$$
$$\text{where} \quad \overrightarrow{\mathbf{x}} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Solution:

1. Let's take the second equation and solve for y:

$$y = x' - x$$

Then plug this into the first equation:

$$\begin{aligned} x' + (x' - x)' &= 2 \\ x' + x'' - x' &= 2 \\ x'' &= 2 \\ \Rightarrow \quad x' &= 2t + c_1 \\ \end{aligned}$$
$$\boxed{x(t) = t^2 + c_1 t + c_2}$$

Then we have:

$$\begin{aligned} y(t) &= x' - x \\ &= \left(t^2 + c_1 t + c_2\right)' - (t^2 + c_1 t + c_2) \\ &= 2t + c_1 - t^2 - c_1 t - c_2 \end{aligned}$$
$$\boxed{y(t) = -t^2 + (2 - c_1)t + c_1 - c_2}$$

2. We first find the eigenvalues of the coefficient matrix A:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(-1-\lambda) - (0)(2) &= 0 \\ (1-\lambda)(-1-\lambda) &= 0 \\ \Rightarrow \quad \lambda = 1,\ \lambda &= -1 \end{aligned}$$

Now find an eigenvector for each eigenvalue. Let's start with $\lambda = 1$. To find an associated eigenvector we solve:

$$
\begin{aligned}
(A - 1I)\overrightarrow{\mathbf{x}} &= \overrightarrow{\mathbf{0}} \\
\begin{bmatrix} 1-1 & 1 \\ 0 & -1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\Rightarrow x_2 = 0,
\end{aligned}
$$

Since x_1 can be anything, we'll just say $x_1 = 1$. Therefore, an eigenvector for $\lambda = 1$ is:

$$
\lambda = 1: \quad \overrightarrow{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

Now let's find an eigenvector for $\lambda = -1$:

$$
\begin{aligned}
(A + 1I)\overrightarrow{\mathbf{x}} &= \overrightarrow{\mathbf{0}} \\
\begin{bmatrix} 1+1 & 1 \\ 0 & -1+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\Rightarrow 2x_1 + x_2 = 0
\end{aligned}
$$

Set $x_2 = 2$, then $x_1 = -1$. Therefore, an eigenvector for $\lambda = -1$ is:

$$
\lambda = -1: \quad \overrightarrow{\mathbf{x}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}
$$

The general solution is then:

$$
\boxed{\overrightarrow{\mathbf{x}} = c_1 e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}
$$