Math 220 Quiz 4 Solution

1. Compute the following expressions:

   (a) $\mathscr{L}\left\{u(t-1) - t^3 + e^t \cos t\right\}$

   (b) $\mathscr{L}^{-1}\left\{\dfrac{3}{(s-1)(s+2)}\right\}$

2. Use the Laplace Transform to find the solution to:

$$y'' + y' - 2y = 3u(t-3),\ y(0) = 0,\ y'(0) = 0$$

Solution:

1. (a) $\mathscr{L}\left\{u(t-1) - t^3 + e^t \cos t\right\} = \mathscr{L}\{u(t-1)\} - \mathscr{L}\{t^3\} + \mathscr{L}\{e^t \cos t\} = \boxed{\dfrac{e^{-s}}{s} - \dfrac{6}{s^4} + \dfrac{s-1}{(s-1)^2+1}}$

   (b) First, use partial fraction decomposition:

$$\frac{3}{(s-1)(s+2)} = \frac{1}{s-1} - \frac{1}{s+2}$$

   Then we have:

$$\mathscr{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+2}\right\} = \boxed{e^t - e^{-2t}}$$

2. Taking the Laplace Transform of the equation we get:

$$\begin{aligned} \mathscr{L}\{y'' + y' - 2y\} &= \mathscr{L}\{3u(t-3)\} \\ \mathscr{L}\{y''\} + \mathscr{L}\{y'\} - 2\mathscr{L}\{y\} &= 3\mathscr{L}\{u(t-3)\} \\ s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) &= \frac{3e^{-3s}}{s} \\ Y(s)(s^2+s-2) &= \frac{3e^{-3s}}{s} \\ Y(s) &= \frac{3e^{-3s}}{s(s^2+s-2)} = e^{-3s}F(s) \end{aligned}$$

   In order to compute $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, we have to use partial fraction decomposition on $F(s)$:

$$\frac{3}{s(s^2+s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$
$$3 = A(s-1)(s+2) + Bs(s+2) + Cs(s-1)$$

   Plugging in $s = 0$ we get $A = -\frac{2}{3}$. Plugging in $s = 1$ we get $B = 1$. Finally, plugging in $s = -2$ we get $C = \frac{1}{2}$. The inverse Laplace Transform of $F(s)$ is:

$$\mathscr{L}^{-1}\left\{\frac{-\frac{2}{3}}{s} + \frac{1}{s-1} + \frac{\frac{1}{2}}{s+2}\right\} = -\frac{2}{3} + e^t + \frac{1}{2}e^{-2t}$$

   Using the property that $\mathscr{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u(t-c)$, the solution is then:

$$y(t) = \mathscr{L}^{-1}\left\{e^{-3s}F(s)\right\} = \boxed{\left(-\frac{2}{3} + e^{t-3} + \frac{1}{2}e^{-2(t-3)}\right)u(t-3)}$$