Math 220 Quiz 5 Solution

1. Find the solution to the following initial value problem:

$$y'' - y = 2\delta(t-3),\ y(0) = 0,\ y'(0) = 0$$

2. Determine the Taylor Polynomial of degree 2 that approximates the solution to:

$$y' = x^2 + e^y,\ y(0) = 0$$

Solution:

1. Start by taking the Laplace transform of the equation and solve for $Y(s)$:

$$\begin{aligned} \mathscr{L}\{y''\} - \mathscr{L}\{y\} &= 2\mathscr{L}\{\delta(t-3)\} \\ s^2Y(s) - sy(0) - y'(0) - Y(s) &= \frac{2e^{-3s}}{s} \\ (s^2-1)Y(s) &= \frac{2e^{-3s}}{s} \\ Y(s) &= \frac{2e^{-3s}}{s(s^2-1)} \end{aligned}$$

The solution is $y(t) = \mathscr{L}^{-1}\{Y(s)\}$. In order to perform this calculation, we must use partial fraction decomposition on the portion of $Y(s)$ that does not include e^{-3s}. Omitting the details, we have:

$$\begin{aligned} F(s) &= \frac{2}{s(s^2-1)} = -\frac{2}{s} + \frac{1}{s-1} + \frac{1}{s+1} \\ \Rightarrow\quad f(t) &= \mathscr{L}^{-1}\{F(s)\} = -2 + e^t + e^{-t} \end{aligned}$$

Using the property that $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$ we have:

$$\boxed{y(t) = \left[-2 + e^{t-3} + e^{-(t-3)}\right]u(t-3)}$$

2. The Taylor polynomial of degree 2 about $x = 0$ is of the form:

$$P_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2$$

We given that $y(0) = 0$. Let's use this and the ODE to compute $y'(0)$:

$$y'(0) = (0)^2 + e^{y(0)} = 0 + e^0 = 1$$

To compute $y''(0)$ we must first compute y'':

$$y'' = (x^2 + e^y)' = 2x + e^y y'$$

Then we have:

$$y''(0) = 2(0) + e^{y(0)}y'(0) = 0 + e^0(1) = 1$$

Therefore, we have:

$$\boxed{y(x) \approx P_2(x) = 0 + x + \frac{1}{2}x^2}$$