Math 220 Quiz 5 Solution

1. Find the solution to the following initial value problem:
$$y'' - 4y = 4\delta(t-1),\ y(0) = 0,\ y'(0) = 0$$
2. Determine the Taylor Polynomial of degree 2 that approximates the solution to:
$$y' = x + y^2,\ y(0) = 1$$

Solution:

1. Start by taking the Laplace transform of the equation and solve for $Y(s)$:
$$\begin{aligned}\mathscr{L}\{y''\} - 4\mathscr{L}\{y\} &= 4\mathscr{L}\{\delta(t-1)\} \\ s^2Y(s) - sy(0) - y'(0) - 4Y(s) &= \frac{4e^{-s}}{s} \\ (s^2-4)Y(s) &= \frac{4e^{-s}}{s} \\ Y(s) &= \frac{4e^{-s}}{s(s^2-4)}\end{aligned}$$
The solution is $y(t) = \mathscr{L}^{-1}\{Y(s)\}$. In order to perform this calculation, we must use partial fraction decomposition on the portion of $Y(s)$ that does not include e^{-s}. Omitting the details, we have:
$$\begin{aligned} F(s) = \frac{4}{s(s^2-4)} &= -\frac{1}{s} + \frac{\frac{1}{2}}{s-2} + \frac{\frac{1}{2}}{s+2} \\ \Rightarrow\quad f(t) = \mathscr{L}^{-1}\{F(s)\} &= -1 + \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}\end{aligned}$$
Using the property that $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$ we have:
$$\boxed{y(t) = \left[-1 + \frac{1}{2}e^{2(t-1)} + e^{-2(t-1)}\right]u(t-1)}$$

2. The Taylor polynomial of degree 2 about $x = 0$ is of the form:
$$P_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2$$
We given that $y(0) = 1$. Let's use this and the ODE to compute $y'(0)$:
$$y'(0) = 0 + [y(0)]^2 = 0 + 1^2 = 1$$
To compute $y''(0)$ we must first compute y'':
$$y'' = (x + y^2)' = 1 + 2yy'$$
Then we have:
$$y''(0) = 1 + 2y(0)y'(0) = 1 + 2(1)(1) = 3$$
Therefore, we have:
$$\boxed{y(x) \approx P_2(x) = 1 + x + \frac{3}{2}x^2}$$