Spring, 2007 – Exam 1 Solutions

1. (20 pts) Consider the initial value problem:

$$y' = 2xy(1+x^2)^{-1/2}, \quad y(0) = 1$$

   (a) Is the differential equation separable?
   (b) Is the differential equation linear?
   (c) State the method you will use to solve for $y(x)$ and then find the solution (you may leave your answer in implicit form).

Solution:

   (a) The differential equation is separable.
   (b) The differential equation is also linear.
   (c) We can use two methods to solve. Let's use Separation of Variables:

$$\begin{aligned} \frac{dy}{dx} &= 2xy(1+x^2)^{-1/2} \\ \frac{dy}{y} &= \frac{2x}{\sqrt{1+x^2}}\,dx \\ \int \frac{dy}{y} &= \int \frac{2x}{\sqrt{1+x^2}}\,dx \\ \ln|y| &= 2\sqrt{1+x^2} + C \end{aligned}$$

   Use the initial condition $y(0) = 1$ to solve for C:

$$\begin{aligned} \ln|1| &= 2\sqrt{1+0^2} + C \\ C &= -2 \end{aligned}$$

   The solution is then:

$$\boxed{\ln|y| = 2\sqrt{1+x^2} - 2}$$

2. (20 pts) Consider the initial value problem: $y' = \dfrac{3}{x} + y$ with $y(1) = -1$.

   (a) Use Euler's method with step size $h = 0.5$ to approximate the solution $y(x)$ at the point $x = 2$.
   (b) Use the improved Euler's method with step size $h = 0.5$ to approximate the solution $y(x)$ at the point $x = 1.5$.

Solution:

   (a) One step of Euler's method gives us the approximate value of $y(1.5)$. We'll call this y_e for use in part (b):

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) = y_0 + h\left(\frac{3}{x_0} + y_0\right) \\ y_1 &= -1 + \frac{1}{2}\left(\frac{3}{1} - 1\right) \\ y_1 &= 0 \\ x_1 &= x_0 + h = 1 + \frac{1}{2} = 1.5 \end{aligned}$$

So, $y_e = 0$. One more step gives us the approximate value of $y(2)$:

$$\begin{aligned}
y_2 &= y_1 + hf(x_1, y_1) = y_1 + h\left(\frac{3}{x_1} + y_1\right) \\
y_2 &= 0 + \frac{1}{2}\left(\frac{3}{1.5} + 0\right) \\
y_2 &= 1 \\
x_2 &= x_1 + h = 1.5 + 0.5 = 2
\end{aligned}$$

Therefore, $\boxed{y(2) \approx 1}$.

(b) One step of the Improved Euler's method gives us the approximate value of $y(1.5)$:

$$\begin{aligned}
y_1 &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] = y_0 + \frac{h}{2}\left[\left(\frac{3}{x_0} + y_0\right) + \left(\frac{3}{x_1} + y_e\right)\right] \\
y_1 &= -1 + \frac{1/2}{2}\left[\left(\frac{3}{1} - 1\right) + \left(\frac{3}{1.5} + 0\right)\right] \\
y_1 &= 0 \\
x_1 &= x_0 + h = 1 + \frac{1}{2} = 1.5
\end{aligned}$$

Therefore, $\boxed{y(1.5) \approx 0}$.

3. (20 pts) Complete each of the following:

(a) Find the general solution to: $y'' + 4y' + 8y = 0$ (your answer should **not** contain the imaginary number i).

(b) Write the form of the particular solution to: $y'' + 4y' + 8y = 1 + e^{-2x}\cos 2x$ (do not solve for the coefficients).

Solution:

(a) The auxiliary equation and its roots are:

$$\begin{aligned}
r^2 + 4r + 8 &= 0 \\
r &= \frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)} \\
r &= -2 \pm 2i
\end{aligned}$$

The general solution is then:

$$\boxed{y = e^{-2x}(c_1\cos 2x + c_2\sin 2x)}$$

(b) The second term in the non-homogeneous part is one of our homogeneous solutions. Therefore, we have a special case. The form of y_p is:

$$\boxed{y_p = A + xe^{-2x}(B\cos 2x + C\sin 2x)}$$

4. (10 pts) A nitric acid solution flows at a constant rate of 4 L/min into a large tank that initially held 100 L of pure water. The solution inside the tank is kept well-stirred and flows out of the tank at a rate of 3 L/min. If the concentration of nitric acid in the solution entering the tank is 0.1, set up but do not solve the initial value problem for $x(t)$, the volume of nitric acid in the tank at time t.

Solution: Using the general equation for mixing problems of this type:

$$\frac{dA}{dt} = r_i c_i - r_o \frac{A}{V_0 + (r_i - r_o)t}$$

and plugging in numbers we get:

$$\boxed{\frac{dA}{dt} = 0.4 - \frac{3}{100+t}A}$$

with the initial condition $\boxed{A(0) = 0}$.

5. (30 pts) Consider the following second order, linear, constant coefficient, non-homogeneous differential equation:

$$y'' + 6y' + 5y = 3e^{-2x}.$$

(a) Use the method of undetermined coefficients to find the particular solution $y_p(x)$ (you must solve for the coefficient(s)).

(b) Find the general solution.

(c) Now use variation of parameters to find the particular solution $y_p(x)$. Note: (1) If you have done this correctly, you will get the same answer as in part (a). (2) The following equations may be helpful:

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= \frac{g(x)}{a} \end{aligned}$$

Solution:

(a) We guess a particular solution and compute its derivatives as follows:

$$\begin{aligned} y_p &= Ae^{-2x} \\ y_p' &= -2Ae^{-2x} \\ y_p'' &= 4Ae^{-2x} \end{aligned}$$

This is not a special case since $r = -2$ is not a root of the auxiliary equation. Plugging y_p and its derivatives into the ODE we get:

$$\begin{aligned} y_p'' + 6y_p' + 5y_p &= 3e^{-2x} \\ 4Ae^{-2x} + 6(-2Ae^{-2x}) + 5Ae^{-2x} &= 3e^{-2x} \\ -3Ae^{-2x} &= 3e^{-2x} \\ -3A &= 3 \\ A &= -1 \end{aligned}$$

The particular solution is then $\boxed{y_p = -e^{-2x}}$.

(b) The general solution is $y = y_h + y_p$. We just found y_p so we must now find y_h. The auxiliary equation and its roots are:

$$\begin{aligned} r^2 + 6r + 5 &= 0 \\ (r+5)(r+1) &= 0 \\ r = -5, \quad r &= -1 \end{aligned}$$

The general solution is then:

$$\boxed{y = c_1 e^{-5x} + c_2 e^{-x} - e^{-2x}}$$

(c) From the homogeneous solution we set $y_1 = e^{-5x}$ and $y_2 = e^{-2x}$. Plugging these into the variation of parameter equations we have:

$$\begin{aligned} v_1' e^{-5x} + v_2' e^{-x} &= 0 \\ -5v_1' e^{-5x} - v_2' e^{-x} &= 3e^{-2x} \end{aligned}$$

Solving for v_1' and v_2' we get:

$$\begin{aligned} v_1' &= -\frac{3}{4}e^{3x} \\ v_2' &= \frac{3}{4}e^{-x} \end{aligned}$$

We get v_1 and v_2 by integrating the above equations:

$$\begin{aligned} v_1 &= -\frac{1}{4}e^{3x} \\ v_2 &= -\frac{3}{4}e^{-x} \end{aligned}$$

The particular solution is then:

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ y_p &= -\frac{1}{4}e^{3x}e^{-5x} - \frac{3}{4}e^{-x}e^{-x} \\ y_p &= -\frac{1}{4}e^{-2x} - \frac{3}{4}e^{-2x} \\ y_p &= -e^{-2x} \end{aligned}$$

just as we obtained in part (a).