Spring, 2007 – Exam 2 Solutions

1. (20 pts) Find the general solution to the following system of first order ODEs:

$$\frac{dx}{dt} = x + 3y + 4t$$
$$\frac{dy}{dt} = x - y$$

Solution: It's easiest to use elimination here. We'll start by solving the second equation for x:

$$x = y' + y$$

Now we plug this result into the first equation and simplify:

$$\begin{aligned} x' &= x + 3y + 4t \\ (y' + y)' &= y' + y + 3y + 4t \\ y'' + y' &= y' + 4y + 4t \\ y'' - 4y &= 4t \end{aligned}$$

The auxiliary equation for the above ODE is $r^2 - 4 = 0$ and its roots are $r = \pm 2$. The homogeneous solution is then:

$$y_h(t) = c_1 e^{2t} + c_2 e^{-2t}$$

The particular solution is of the form $y_p(t) = At + B$. When we plug this into the ODE and solve for A and B we find that $A = -1$ and $B = 0$. Therefore, the particular solution is $y_p = -t$ and the general solution is:

$$\boxed{y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^{-2t} - t}$$

The solution for $x(t)$ is obtained from the equation $x(t) = y'(t) + y(t)$ and goes as follows:

$$\boxed{x(t) = 3c_1 e^{2t} - c_2 e^{-2t} - t - 1}$$

2. (20 pts) Compute the following expressions:

   (a) $\mathscr{L}\left[t + \sin 3t + e^{2t}\cos t\right]$

   (b) $\mathscr{L}\left[te^t\right]$

   (c) $\mathscr{L}^{-1}\left[\dfrac{2}{s(s^2+4)}\right]$

Solution:

(a) $\mathscr{L}\left[t + \sin 3t + e^{2t}\cos t\right] = \mathscr{L}[t] + \mathscr{L}[\sin 3t] + \mathscr{L}[e^{2t}\cos t] = \boxed{\dfrac{1}{s^2} + \dfrac{3}{s^2+9} + \dfrac{s-2}{(s-2)^2+1}}$

(b) $\mathscr{L}\left[te^t\right] = \boxed{\dfrac{1}{(s-1)^2}}$

(c) Let $F(s) = \frac{1}{s}$ and $G(s) = \frac{2}{s^2+4}$. Then we have $f(t) = \mathscr{L}^{-1}[F(s)] = 1$ and $g(t) = \mathscr{L}^{-1}[G(s)] = \sin 2t$. Using the convolution property we find that:

$$\begin{aligned}
\mathscr{L}^{-1}[F(s)G(s)] &= f(t) * g(t) \\
&= \int_0^t 1 \cdot \sin 2v \, dv \\
&= \left[-\frac{1}{2}\cos 2v\right]_0^t \\
&= \boxed{\frac{1}{2} - \frac{1}{2}\cos 2t}
\end{aligned}$$

OR

We could use partial fraction decomposition:

$$\begin{aligned}
\frac{2}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} \\
2 &= A(s^2+4) + (Bs+C)s
\end{aligned}$$

Letting $s = 0$ we find that $A = \frac{1}{2}$. Letting $s = 1$ and $s = -1$ we get the following system of equations:

$$\begin{aligned}
2 &= \frac{5}{2} + B + C \\
2 &= \frac{5}{2} + B - C
\end{aligned}$$

The solution to this system is $B = -\frac{1}{2}$ and $C = 0$. Therefore, we have:

$$\mathscr{L}^{-1}\left[\frac{2}{s(s^2+4)}\right] = \mathscr{L}^{-1}\left[\frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}s}{s^2+4}\right] = \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] = \boxed{\frac{1}{2} - \frac{1}{2}\cos 2t}$$

3. (20 pts) Complete each part below:

(a) Find the function $f(t)$ such that $f(t) = \mathscr{L}^{-1}\left[\frac{2e^{-2s} - 4e^{-4s}}{s}\right]$.

(b) Solve the initial value problem:

$$x'' = f(t), \quad x(0) = 0, \quad x'(0) = 1$$

where $f(t)$ is the function you found in part (a).

Solution:

(a) $\boxed{f(t) = 2u(t-2) - 4u(t-4)}$

(b) We take the Laplace transform and solve for $X(s)$:

$$\begin{aligned}
x'' &= f(t) \\
\mathscr{L}[x''] &= \mathscr{L}[f(t)] \\
s^2X(s) - sx(0) - x'(0) &= \frac{2e^{-2s} - 4e^{-4s}}{s} \\
s^2X(s) - 1 &= \frac{2e^{-2s} - 4e^{-4s}}{s} \\
X(s) &= \frac{2e^{-2s}}{s^3} - \frac{4e^{-4s}}{s^3} + \frac{1}{s^2}
\end{aligned}$$

The solution $x(t)$ is then the inverse Laplace transform of $X(s)$:

$$x(t) = \mathscr{L}^{-1}[X(s)]$$
$$x(t) = \mathscr{L}^{-1}\left[\frac{2e^{-2s}}{s^3}\right] - \mathscr{L}^{-1}\left[\frac{4e^{-4s}}{s^3}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2}\right]$$
$$\boxed{x(t) = (t-2)^2u(t-2) - 2(t-4)^2u(t-4) + t}$$

4. (20 pts) Solve the initial value problem:

$$y'' - 4y = \delta(t-1), \quad y(0), \ y'(0) = 0$$

Solution: We start by taking the Laplace transform of the ODE and solving for $Y(s)$:

$$y'' - 4y = 4\delta(t-1)$$
$$\mathscr{L}[y'' - 4y] = \mathscr{L}[4\delta(t-1)]$$
$$\mathscr{L}[y''] - 4\mathscr{L}[y] = 4\mathscr{L}[\delta(t-1)]$$
$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = 4e^{-s}$$
$$Y(s)(s^2-4) = 4e^{-s}$$
$$Y(s) = \frac{4e^{-s}}{s^2-4}$$

We must now find the inverse Laplace transform of $Y(s)$. To do this, we will use the property that:

$$f(t-a)u(t-a) = \mathscr{L}^{-1}[e^{-as}F(s)]$$

Here, $a = 1$ and $F(s) = \dfrac{4}{s^2-4}$. To find $f(t) = \mathscr{L}^{-1}[F(s)]$, we must perform partial fraction decomposition:

$$\frac{4}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2}$$
$$4 = A(s+2) + B(s-2)$$

Letting $s = 2$ we find that $A = 1$ and letting $s = -2$ we find that $B = -1$. Therefore, we have

$$f(t) = \mathscr{L}^{-1}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] = e^{2t} - e^{-2t}$$

Using the property with $a = 1$ we then get the solution:

$$\boxed{y(t) = f(t-1)u(t-1) = \left(e^{2(t-1)} - e^{-2(t-1)}\right)u(t-1)}$$

5. (20 pts) Find the general solution for each of the following:

   (a) $x^2y'' - 2y = 0$

   (b) $x^2y'' + 5xy' + 4y = 0$

Solution:

(a) The indicial equation and its roots are:

$$\begin{aligned} r(r-1)-2 &= 0 \\ r^2-r-2 &= 0 \\ (r-2)(r+1) &= 0 \\ r = 2, \quad r &= -1 \end{aligned}$$

The roots are real and distinct. Therefore, the general solution is:

$$\boxed{y(x) = c_1 x^2 + c_2 x^{-1}}$$

(b) The indicial equation and its roots are:

$$\begin{aligned} r(r-1)+5r+4 &= 0 \\ r^2+4r+4 &= 0 \\ (r+2)^2 &= 0 \\ r &= -2 \end{aligned}$$

The root $r = -2$ is repeated. Therefore, the general solution is:

$$\boxed{y(x) = c_1 x^{-2} + c_2 x^{-2} \ln x}$$