Math 220 – Section 10.2 Solutions

1. Solve the boundary value problem:

$$y'' - y = 0, \quad y(0) = 0, \quad y(1) = -4$$

The general solution to the differential equation is:

$$y(x) = c_1 e^x + c_2 e^{-x}$$

Using the boundary conditions, we have:

$$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y(1) &= c_1 e + c_2 e^{-1} = -4 \end{aligned}$$

From the first equation, we have $c_2 = -c_1$. Substituting into the second equation and solving for c_1 we have

$$\begin{aligned} c_1 e + c_2 e^{-1} &= -4 \\ c_1 e - c_1 e^{-1} &= -4 \\ c_1(e - e^{-1}) &= -4 \\ c_1 &= -\frac{4}{e - e^{-1}} \end{aligned}$$

Then $c_2 = -c_1 = \dfrac{4}{e - e^{-1}}$ and

$$\boxed{y(x) = \frac{4}{e - e^{-1}} \left(e^{-x} - e^x\right)}$$

7. Solve the boundary value problem:

$$y'' + y = 0, \quad y(0) = 1, \quad y(2\pi) = 1$$

The general solution to the differential equation is:

$$y(x) = c_1 \sin x + c_2 \cos x$$

Using the boundary conditions, we have:

$$\begin{aligned} y(0) &= c_2 = 1 \\ y(2\pi) &= c_2 = 1 \end{aligned}$$

The boundary conditions don't tell us the value of c_1. Therefore, it can be anything. Thus, the solution is

$$\boxed{y(x) = c_1 \sin x}$$

9. Find the values of λ for which the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$$

has a nontrivial solution and determine the corresponding nontrivial solutions.

(a) Assume that $\lambda > 0$. The general solution to the differential equation is

$$y(x) = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x)$$

Using the boundary conditions, we have:

$$\begin{aligned} y(0) &= c_2 = 0 \\ y'(\pi) &= -\sqrt{\lambda} c_1 \cos(\sqrt{\lambda}\pi) = 0 \end{aligned}$$

To avoid the nontrivial solution, we must have $\cos(\sqrt{\lambda}\pi) = 0$. This happens when

$$\begin{aligned} \sqrt{\lambda}\pi &= \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5\pi}{2}, \ \ldots \\ \sqrt{\lambda} &= \frac{1}{2}, \ \frac{3}{2}, \ \frac{5}{2}, \ \ldots \\ \lambda &= \frac{1}{4}, \ \frac{9}{4}, \ \frac{25}{4}, \ \ldots \\ \lambda &= \frac{(2n-1)^2}{4}, \quad n = 1, 2, 3, \ldots \end{aligned}$$

The corresponding nontrivial solutions are:

$$\boxed{y_n(x) = c_n \sin\left(\frac{2n-1}{2} x\right)}$$

where the c_n can be anything.

(b) Assume that $\lambda = 0$. The general solution to the differential equation is

$$y(x) = c_1 x + c_2$$

Using the boundary conditions, we have:

$$\begin{aligned} y(0) &= c_2 = 0 \\ y'(\pi) &= c_1 = 0 \end{aligned}$$

Therefore, the solution is $y(x) = 0$.

(c) Assume that $\lambda < 0$. The general solution to the differential equation is

$$y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

Using the boundary conditions, we have:

$$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(\pi) &= \sqrt{-\lambda} c_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda} c_2 e^{-\sqrt{-\lambda}} = 0 \end{aligned}$$

The solution to the above system is $c_1 = c_2 = 0$ and the solution to the boundary value problem is $y(x) = 0$.

15. Solve the heat equation with $\beta = 3$, $L = \pi$, and $f(x) = \sin x - 6 \sin 4x$. Plugging in $\beta = 3$ and $L = \pi$ we have:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin(nx)$$

From the initial condition $u(x,0) = f(x)$ we have:

$$u(x,0) = f(x)$$
$$\sum_{n=1}^{\infty} c_n \sin(nx) = \sin x - 6\sin 4x$$
$$c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x + c_4 \sin 4x + c_5 \sin 5x + \ldots = \sin x - 6\sin 4x$$

Equating coefficients on both sides of the equation, we have:

$$c_1 = 1,\ c_2 = 0,\ c_3 = 0,\ c_4 = -6,\ c_n = 0 \text{ for } n \geq 5$$

Therefore, the solution is:

$$\boxed{u(x,t) = e^{-3t}\sin x - 6e^{-48t}\sin 4x}$$

19. Solve the wave equation with $\alpha = 3$, $L = \pi$, $f(x) = 3\sin 2x + 12\sin 13x$, and $g(x) = 0$. Plugging in $\alpha = 3$ and $L = \pi$ we have:

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(3nt) + b_n \sin(3nt)] \sin(nx)$$

From the initial condition $u(x,0) = f(x)$ we have:

$$u(x,0) = f(x)$$
$$\sum_{n=1}^{\infty} a_n \sin(nx) = 3\sin 2x + 12\sin 13x$$
$$a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots + a_{13} \sin 13x + \ldots = 3\sin 2x + 12\sin 13x$$

Equating coefficients on both sides of the equation, we have:

$$a_1 = 0,\ a_2 = 3,\ a_3 = 0,\ \ldots,\ a_{13} = 12,\ a_n = 0 \text{ for } n \geq 13$$

From the initial condition $u_t(x,0) = g(x)$ we have:

$$u_t(x,0) = g(x)$$
$$\sum_{n=1}^{\infty} 3nb_n \sin(nx) = 0$$
$$\Rightarrow \quad b_n = 0 \text{ for } n \geq 1$$

Therefore, the solution is:

$$\boxed{u(x,t) = 3\cos(6t)\sin(2x) + 12\cos(36t)\sin(12x)}$$