1. Solve the boundary value problem:

\[ y'' - y = 0, \quad y(0) = 0, \quad y(1) = -4 \]

The general solution to the differential equation is:

\[ y(x) = c_1 e^x + c_2 e^{-x} \]

Using the boundary conditions, we have:

\[ y(0) = c_1 + c_2 = 0 \]
\[ y(1) = c_1 e + c_2 e^{-1} = -4 \]

From the first equation, we have \( c_2 = -c_1 \). Substituting into the second equation and solving for \( c_1 \) we have

\[ c_1 e + c_2 e^{-1} = -4 \]
\[ c_1 e - c_1 e^{-1} = -4 \]
\[ c_1 (e - e^{-1}) = -4 \]
\[ c_1 = -\frac{4}{e - e^{-1}} \]

Then \( c_2 = -c_1 = \frac{4}{e - e^{-1}} \) and

\[ y(x) = \frac{4}{e - e^{-1}} (e^{-x} - e^x) \]

7. Solve the boundary value problem:

\[ y'' + y = 0, \quad y(0) = 1, \quad y(2\pi) = 1 \]

The general solution to the differential equation is:

\[ y(x) = c_1 \sin x + c_2 \cos x \]

Using the boundary conditions, we have:

\[ y(0) = c_2 = 1 \]
\[ y(2\pi) = c_2 = 1 \]

The boundary conditions don’t tell us the value of \( c_1 \). Therefore, it can be anything. Thus, the solution is

\[ y(x) = c_1 \sin x \]

9. Find the values of \( \lambda \) for which the boundary value problem

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0 \]

has a nontrivial solution and determine the corresponding nontrivial solutions.
(a) Assume that $\lambda > 0$. The general solution to the differential equation is

$$y(x) = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x)$$

Using the boundary conditions, we have:

$$y(0) = c_2 = 0$$
$$y'(\pi) = -\sqrt{\lambda}c_1 \cos(\sqrt{\lambda}\pi) = 0$$

To avoid the nontrivial solution, we must have $\cos(\sqrt{\lambda}\pi) = 0$. This happens when

$$\sqrt{\lambda}\pi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$
$$\sqrt{\lambda} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$
$$\lambda = \frac{1}{4}, \frac{9}{4}, \frac{25}{4}, \ldots$$
$$\lambda = \frac{(2n - 1)^2}{4}, \ n = 1, 2, 3, \ldots$$

The corresponding nontrivial solutions are:

$$y_n(x) = c_n \sin\left(\frac{2n - 1}{2}x\right)$$

where the $c_n$ can be anything.

(b) Assume that $\lambda = 0$. The general solution to the differential equation is

$$y(x) = c_1 x + c_2$$

Using the boundary conditions, we have:

$$y(0) = c_2 = 0$$
$$y'(\pi) = c_1 = 0$$

Therefore, the solution is $y(x) = 0$.

(c) Assume that $\lambda < 0$. The general solution to the differential equation is

$$y(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Using the boundary conditions, we have:

$$y(0) = c_1 + c_2 = 0$$
$$y'(\pi) = \sqrt{-\lambda}c_1 e^{\sqrt{\lambda}\pi} - \sqrt{-\lambda}c_2 e^{-\sqrt{\lambda}\pi} = 0$$

The solution to the above system is $c_1 = c_2 = 0$ and the solution to the boundary value problem is $y(x) = 0$.

15. Solve the heat equation with $\beta = 3$, $L = \pi$, and $f(x) = \sin x - 6 \sin 4x$. Plugging in $\beta = 3$ and $L = \pi$ we have:

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin(nx)$$
From the initial condition \( u(x, 0) = f(x) \) we have:

\[
\begin{align*}
u(x, 0) &= f(x) \\
\sum_{n=1}^{\infty} c_n \sin(nx) &= \sin x - 6 \sin 4x \\
c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x + c_4 \sin 4x + c_5 \sin 5x + \ldots &= \sin x - 6 \sin 4x
\end{align*}
\]

Equating coefficients on both sides of the equation, we have:

\( c_1 = 1, \ c_2 = 0, \ c_3 = 0, \ c_4 = -6, \ c_n = 0 \) for \( n \geq 5 \)

Therefore, the solution is:

\[
\boxed{u(x, t) = e^{-3t} \sin x - 6e^{-48t} \sin 4x}
\]

19. Solve the wave equation with \( \alpha = 3, \ L = \pi, \ f(x) = 3 \sin 2x + 12 \sin 13x, \) and \( g(x) = 0. \) Plugging in \( \alpha = 3 \) and \( L = \pi \) we have:

\[
u(x, t) = \sum_{n=1}^{\infty} [a_n \cos(3nt) + b_n \sin(3nt)] \sin(nx)
\]

From the initial condition \( u(x, 0) = f(x) \) we have:

\[
\begin{align*}
u(x, 0) &= f(x) \\
\sum_{n=1}^{\infty} a_n \sin(nx) &= 3 \sin 2x + 12 \sin 13x \\
a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots + a_{13} \sin 13x + \ldots &= 3 \sin 2x + 12 \sin 13x
\end{align*}
\]

Equating coefficients on both sides of the equation, we have:

\( a_1 = 0, \ a_2 = 3, \ a_3 = 0, \ldots, \ a_{13} = 12, \ a_n = 0 \) for \( n \geq 13 \)

From the initial condition \( u_t(x, 0) = g(x) \) we have:

\[
\begin{align*}
u_t(x, 0) &= g(x) \\
\sum_{n=1}^{\infty} 3nb_n \sin(nx) &= 0 \\
\Rightarrow \quad b_n &= 0 \text{ for } n \geq 1
\end{align*}
\]

Therefore, the solution is:

\[
\boxed{u(x, t) = 3 \cos(6t) \sin(2x) + 12 \cos(36t) \sin(12x)}
\]